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LECTURE NOTES

ON

Mechanics, Sound and Light

BY

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DESIGNED FOR STUDENTS IN THE FIRST HALF-YEAR'S COURSE IN
PHYSICS IN THE UNIVERSITY.

ANN ARBOR, MICH.:
GEORGE WAHR, PUBLISHER.
1890.

[illegible]

THE COURIER OFFICE, PRINTERS.

PREFATORY NOTE.

The following pages include most of the topics treated in the first Course in Physics in the University. They serve merely the purpose of texts for lectures. No attempt has been made to secure fullness nor to indicate the illustrations accompanying the lectures. Such topics have been selected as appear most important, and an effort has been made to put them in logical relation to one another. The matter here in print will relieve the student of a large part of the labor of taking notes; and, it is hoped, will secure for him more accurate statements than he would otherwise obtain.

EXPLANATION OF REFERENCES.

The numerals refer to pages.

The letters, enclosed in brackets accompanying the headings of sections, have the following significations:

A. & B.—Anthony and Brackett's Text-Book on Physics; for Mechanics, Part I.; for Sound and Light, the Complete Work.

D.—Daniell's Text-Book of the Principles of Physics. First edition.

Desch.—Deschanel's Natural Philosophy. Second edition.

G.—Ganot's Physics. Thirteenth edition.

M. & M.—Maxwell's Matter and Motion.

T. & T.—Thomson and Tait's Elements of Natural Philosophy.

T.—Tait's Light.

Several of the cuts in Sound and Light have been kindly furnished by the publishers of Anthony and Brackett's Physics, with the permission of the authors.

MECHANICS.

INTRODUCTORY.

1. Physics Defined.

Physics may be defined as the science of matter and energy.

Natural Science in general is a classified and correlated knowledge of nature in so far as it relates to *matter* and *energy*.

Natural Science includes

1. *Natural History*, which comprises a study of the external forms, the internal structure of bodies, and the functions of their various parts.
2. *Physical Science*, which treats of the matter composing bodies and the phenomena due to the external and internal forces to which they are subjected.

Physical Science comprises

a. *Chemistry*, which has to do with the atomic constitution of bodies and with properties depending upon atomic conditions.

b. *Physics*, which studies the molecular aggregation of matter, the phenomena dependent upon the motion of masses and molecules, the forces producing motion, and the energy associated with all natural phenomena.

2. Mechanics Defined.

The science which investigates the action of *force* is called by most logical writers *Dynamics*. It is commonly called *Mechanics*, a term employed by Newton to designate the science of machines and the art of making them (T. & T. p. 1). Since force is known only by the motion it produces, a discussion of the laws of pure motion should precede that of the laws of force. This constitutes the subject matter of *Kinematics*. To the idea of space, involved in Geometry, is added that of time.

Dynamics is divided in accordance with the two methods in which force is recognized as acting, namely:

a. As preventing motion or change of motion.

b. As producing or changing motion.

Dynamics, therefore, includes

Statics, in which equilibrium under the action of two or more forces is considered.

Kinetics, in which the relation of forces to motion is investigated.

Jan 23 98

3. FUNDAMENTAL UNITS OF MEASUREMENT.

Modern physics is quantitative in its character. The correct determination of the relations of connected phenomena involves the measurement of those relations.

Since all the phenomena of nature occur in *matter*, and are presented to us in *time* and *space* relations, the measurement of physical relations involves the choosing of three fundamental units, viz: the unit of *length*, the unit of *mass*, and the unit of *time*.

All other units employed in Physics are expressible in terms of those of length, mass, and time. They are, therefore, called *derived* units in distinction from the other three, which are called *fundamental* units.

In the absolute or C. G. S. system the unit of length is the centimeter; of mass, the gramme; and of time, the second.

The *centimeter* is the hundredth part of the length of a bar of platinum preserved in the Archives of Paris. It must be taken at the temperature of melting ice. Its value in eight different feet, which have been in use in Europe, is given in the table.

One Meter	Paris Feet	Prussian Feet	English Feet	Austrian Feet	Bavarian Feet	Baden Feet	Saxon Feet	Swedish Feet
=	3 078444	3.186109	3.280856	3.163446	3.426310	3.333333	3.531197	3.368126

The meter was intended to be the one ten-millionth part of an earth-quadrant from the equator to the pole. It is now known that such a quadrant equals about 10,000,880 meters. This difference between the *ideal* and the *legal* meter illustrates the difference between a *theoretical* and a *practical* unit.

The unit of mass is the *gramme*, originally defined as the mass of a cubic centimeter of distilled water at maximum density. It is now practically defined as the $\frac{1}{1000}$ part of a platinum weight preserved in the Archives of Paris. This weight is called a kilogramme.

The unit of *time* is the *second*, or the $\frac{1}{86400}$ part of a *mean solar day*.

KINEMATICS.

4. Motion. (M. & M., 38.)

The configuration of a system of points or bodies is their relative positions. When a change of configuration is considered only with respect to its state at the beginning and end of the process of change, without respect to time, it is called *displacement*.

But when the attention is directed to the process itself, as taking place within a certain period and in a continuous manner, the change of configuration is called *motion*.

Motion is the change in the relative position of a material particle. All motion is relative.

5. PATH.

The path of a moving body must always be continuous. There can be no abrupt change of velocity nor of direction of motion, since such abrupt change would imply the action of an infinite force. If we select any point of the path, the particle will pass that point at least once during the motion. Mathematical curves may be discontinuous, but a material particle cannot traverse a discontinuous path while retaining continuous existence in time and space.

6. DIRECTION OF MOTION AND CURVATURE. (T. & T., 2.)

The direction of motion at any point of a curved path is the tangent to the curve at that point. If the path be not straight the direction of motion changes from point to point; the *rate* of this change, per unit length of the curve, is called the *curvature*.

Let the curvature between the points P and Q, Fig. 1, be considered. Let θ be the angle between the radii P O and Q O, and let s be the arc P Q. The curvature is the ratio of θ to s, since θ is the angle between the tangents at P and Q. Then if r be the radius.

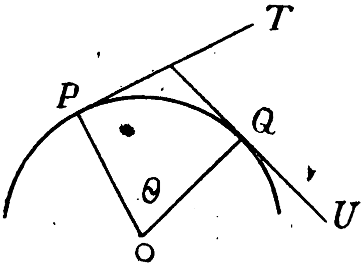


FIGURE 1.

$$r\,\theta = s, \text{ or } \frac{\theta}{s} = \frac{1}{r} \quad \dots\dots\dots\dots (1)$$

The first member of the equation is by definition the curvature. The curvature is therefore equal to the reciprocal of the radius.

7. SPEED AND VELOCITY.

Rate of motion, without reference to direction, is called *speed*. But when the motion is a directed quantity, that is, along a line whose direction is given, then the rate of motion is called *velocity.*

Velocity is either *uniform* or *variable.* Constant velocity is measured by the ratio of the path traversed to the time of traversing it; or

$$v = \frac{s}{t} \dots\dots\dots\dots\dots\dots (2)$$

The practical scientific unit of velocity is the velocity of one cm. per second.

When the velocity is variable, the velocity at any instant may be defined as the space which the particle would describe if it should move uniformly for a second from that instant with velocity unchanged.

8. ACCELERATION.

Acceleration is the rate of change of velocity. It may be either positive or negative, a negative acceleration being a retardation. Acceleration is either uniform or variable. Uniform acceleration is measured by the change of velocity taking place during a unit of time. The practical unit of acceleration is an acceleration of one cm. per second.

The formula for uniform acceleration is

$$f = \frac{v - v_0}{t} \dots\dots\dots\dots\dots\dots (3)$$

in which v_0 and v are the initial and final velocities for time t.

9. SPACE DESCRIBED IN UNIFORMLY ACCELERATED MOTION.

The equivalent velocity, or that with which a uniformly moving particle would describe the same space in the same time, is equal to the half sum of the initial and final velocities.

Hence

$$s = \frac{v + v_0}{2} t \dots\dots\dots\dots\dots (4)$$

But $$v = v_0 + f\,t \qquad (5)$$

Substituting in (4)

$$s = \frac{2\,v_0 + f\,t}{2}\,t = v_0\,t + \tfrac{1}{2}\,f\,t^2 \qquad (6)$$

Multiply (3) by (4) and

$$f\,s = \frac{v^2 - {v_0}^2}{2}$$

$$\text{or } v^2 = {v_0}^2 + 2\,f\,s \qquad (7)$$

If the initial velocity is zero, then (5), (6) and (7) become

$$v = f\,t \qquad (8)$$
$$s = \tfrac{1}{2}\,f\,t^2 \qquad (9)$$
$$v^2 = 2\,f\,s \qquad (10)$$

When t is unity (9) becomes

$$s = \tfrac{1}{2}\,f,$$

or space described in unit time is half the acceleration. If the particle start from rest, the velocity at the end of the first second is the acceleration. Hence the space described in the first second from rest is half the velocity attained.

10. SECOND METHOD OF OBTAINING SPACE IN UNIFORMLY ACCELERATED MOTION.

Let the whole time be divided into a very large number n of equal intervals τ.

Then $n\,\tau = t$.

The velocities at the end of the several small time intervals are

$$f\,\tau,\ 2\,f\,\tau,\ 3\,f\,\tau\ \ldots\ldots\ldots\ n\,f\,\tau.$$

If we suppose the velocities constant during these small intervals, then the spaces described in the successive intervals are

$$f\tau^2,\ 2\,f\,\tau^2,\ 3\,f\,\tau^2 \,\ldots\ldots\ldots\ldots\ldots\ldots\, n\,f\,\tau^2.$$

The entire space is the sum of these elements, or

for " 9 "

$$s = f\,\tau^2\,(1 + 2 + 3 + \ldots\ldots\ldots\ldots\ldots\ldots n).$$

But when *n* is very large the sum of the m^{th} powers of the natural numbers 1, 2, 3, &c, to n is

$$s = \frac{n^{m+1}}{m+1}.$$

Therefore $1 + 2 + 3 \,\ldots\ldots\ldots\ldots\, n = \frac{n^2}{2}$, and

$$s = \tfrac{1}{2}\,f\,\tau^2\,n^2 = \tfrac{1}{2}\,f\,t^2, \text{ as before.}$$

NOTE.

To find the sum of s of the series

$$1^m + 2^m + 3^m + \ldots\ldots\ldots\ldots\ldots\ldots n^m.$$

We have (Todhunter's Algebra, p. 405) [illegible]

$$s = Cn^{m+1} + A_0 n^{m} + \tfrac{m}{2} A_1\, n^{m-1} + \tfrac{m(m-1)}{2\,.\,3} A_2\, n^{m-2} + \&c.,$$

in which $C = \frac{1}{m+1}$; $A_0 = \frac{1}{2}$; $A_1 = \frac{1}{6}$; $A_2 = o$ &c.

All coefficients with even subscripts equal zero.

Hence

$$s = \frac{n^{m+1}}{m+1} + \tfrac{1}{2}\,n^m + \tfrac{m}{12}\,n^{m-1} - \frac{m(m-1)\,(m-2)}{2\,.\,3\,.\,4\,.\,30}\,n^{m-3}\ \&c.$$

If we suppose the series cleared of fractions,

$$a\,s = n^{m+1} + b\,n^m + cn^{m-1} - \&c.$$

Omitting constants, s is of the form

$$s = n^{m+1} + n^m + n^{m-1} - \&c.$$

But $n^{m+1} = n^m \times n$, and $n^{m+1} + n^m = n^m(n+1)$.

If now *n* is *very large*, then unity may be omitted in comparison with it, or

$$n^{m+1} + n^m = n^{m+1}.$$

Hence, the first two terms equal the first term when n is very large, or the second term may be omitted in comparison with the first. The succeeded terms are all still lower powers of n, and are therefore negligible in comparison with the first term. Hence

$$s = \frac{n^{m+1}}{m+1}.$$

11. THE FREE FALL OF BODIES.

The acceleration due to gravity is denoted by g. Since this acceleration is a constant at any one place we may substitute g for f in the preceding general formulæ.

Hence

$$\left.\begin{aligned} v &= g\,t \\ s &= \tfrac{1}{2}\,g\,t^2 \\ v^2 &= 2\,g\,s \end{aligned}\right\} \quad (11)$$

If the body starts with an initial velocity, then

$$\left.\begin{aligned} v &= v_0 \pm g\,t \\ s &= v_0 t \pm \tfrac{1}{2}\,g\,t^2 \\ v^2 &= v_0^2 \pm 2\,g\,s \end{aligned}\right\} \quad (12)$$

12. EXPERIMENTAL DEMONSTRATION OF THE LAWS OF FALLING BODIES.

Atwood's Machine. The Chronoscope.
(Explained in lectures.)

13. PROJECTION UPWARD.

If motion upward is reckoned positive, then the acceleration is negative and equations (12) apply with the minus sign. Then

$$\begin{aligned} v &= v_0 - g\,t, \\ s &= v_0\,t - \tfrac{1}{2}\,g\,t^2, \\ v^2 &= v_0^2 - 2\,g\,s. \end{aligned}$$

a. To find time of ascent. When the body reaches its highest point $v=0$.

$$\text{Hence } v_0 = g\,t \text{ and } t = \frac{v_0}{g}.$$

b. To find height to which the body will ascend. Since $v=o$ at highest point, $v_0^2=2\,g\,s$ and $s=\frac{v_0^2}{2g}$. Neglecting atmospheric resistance, the body returns to the point of projection with the initial velocity in the opposite direction.

c. To find time when the body will be at a given height.

Substitute the given height for *s* in the second equation. This gives an equation of the second degree with two roots. The interpretation is that the body will be at the given height once while ascending and again when descending.

14. COMPOSITION OF MOTIONS, VELOCITIES, AND ACCELERATIONS. (A. & B., 19; T. & T., 8.)

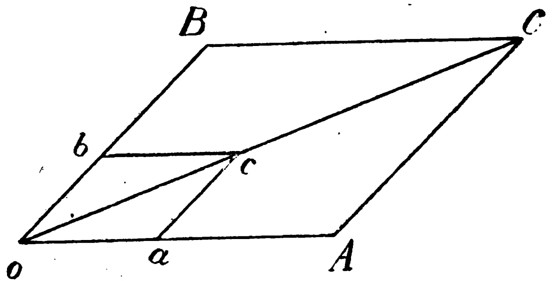


FIGURE 2.

To compound any two velocities as O A and O B, Fig. 2, complete the parallelogram and draw the diagonal O C. Then will O C represent the resultant of O A and O B both in magnitude and direction. If two velocities, which are given to a point, are represented by two adjacent sides of a parallelogram, then the resultant velocity will be represented by the diagonal of the parallelogram drawn through the intersection of these two sides. Accelerations and simple motions may be compounded in the same way as velocities.

15. VELOCITIES RESOLVED IN ANY TWO RECTANGULAR DIRECTIONS.

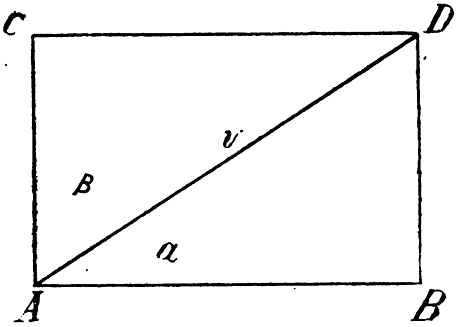


FIGURE 3.

Let A D be *v*, the velocity to be resolved. Let A C and AB, Fig. 3, be the two rectangular directions. Then the two components required are A B and A C.

$$\text{But } A\ B = v \cos \alpha;$$
$$A\ C = v \cos \beta = v \sin \alpha.$$

Hence to find the component in any direction multiply by the cosine of the "direction angle."

16. EXTENSION OF THE PARALLELOGRAM OF VELOCITIES.

(Expanded in lectures.)

17. PATH OF A PROJECTILE. (A. & B., 84.)

Let *v* be the velocity of projection in the direction O Y, Fig. 4. This motion must be compounded with an acceleration *g* in a vertical direction. For the two motions in time *t* we have

$$y = vt,$$
$$x = \tfrac{1}{2}\,gt^2.$$

Whence $y^2 = 2\,\frac{v^2x}{g}$, an equation of the form $y^2 = 2px$ which is the equation of a *parabola*.

Or otherwise: O A : O B : : t : t′ (along O Y);
O C : O D : : t^2 : t'^2 (along O X).

Whence O C : O D : : O A^2 : O B^2; or the ordinates are proportional to the squares of the abscissas, a property characteristic of the parabola.

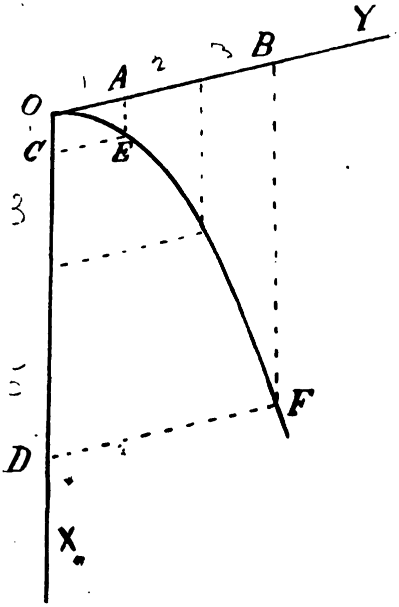


FIGURE 4.

18. Motion on an Inclined Plane.

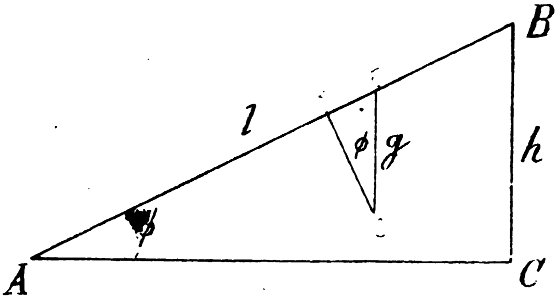


FIGURE 5.

Let φ be the angle of elevation of the plane. Resolve the acceleration *g* into two components, one normal to the plane and the other along the plane. The former equals g cos φ and is ineffective in producing motion; the other equals g sin φ. Substituting this acceleration for *f* in equations (8), (9), (10), we have

$$\left.\begin{aligned} v &= g \sin \varphi t \\ s &= \tfrac{1}{2} g \sin \varphi t^2 \\ v^2 &= 2\, g \sin \varphi s \end{aligned}\right\} \quad (13)$$

Since $\sin \varphi = \frac{h}{l}$, $v^2 = 2g \frac{h}{l} l = 2gh$, when the body falls the entire length of the plane. The velocity in falling down the plane is the same as in falling down the vertical height of the plane.

From the second equation

$$l = \tfrac{1}{2} g \frac{h}{l} t^2, \text{ or } t^2 = \frac{2l^2}{gh}.$$

Whence *t* varies as *l* if *h* remains constant.

19. UNIFORM CIRCULAR MOTION. (T. & T., 9.)

Acceleration should now be extended to mean rate of change of velocity in any direction, whether in the direction of the motion or not. In uniform circular motion the velocity of the point is measured along the circumference, while it is required to find the constant acceleration toward the center. Let *v* be the velocity along the circle. Then if A D, Fig. 6, be traversed in time *t*,

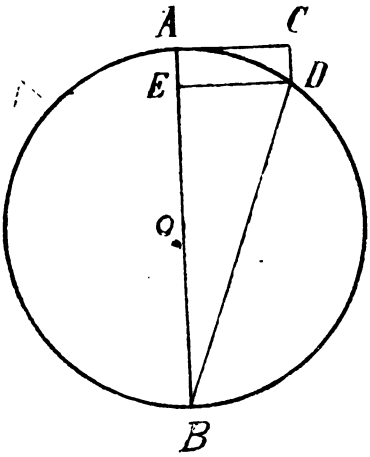


FIGURE 6.

$$A\,D = vt.$$

The change of motion from A C to A D is AE. Since the centripetal acceleration is uniform, $AE = \frac{1}{2} ft^2$, in which f is the required acceleration. If A D is very small,

$$AE : AD :: AD : AB;$$
$$\text{and } AD^2 = AE \times AB.$$

Whence $v^2t^2 = \frac{1}{2} ft^2 \times 2r$ (r being radius of circle). From this equation, $f = \frac{v^2}{r}$.

If T is the time of a complete revolution of the point, $v = \frac{2\pi r}{T}$ and $v^2 = \frac{4\pi^2 r^2}{T^2}$.

Substituting, $f = \frac{4\pi^2 r}{T^2}$.

20. SIMPLE HARMONIC MOTION. (T. & T., 19; Desch., 64; D., 79; A. & B., 21; Everett's Vibratory Motion and Sound.)

Simple Harmonic Motion (denoted by S. H. M.) is the apparent motion of a point, moving with uniform velocity round a circle, when the observer is in the plane of the circle and at a great distance. This apparent motion is along a straight line with a maximum velocity at its middle point.

S. H. M. may be defined as the motion of a point along a straight line with an acceleration constantly directed toward the middle point of the line, and varying in magnitude directly as the displacement from that middle point.

The circle is called the *auxiliary circle* or *circle of reference*, and its radius is the *amplitude* of the S. H. M.

The *period* of the S. H. M. is the time of a complete revolution of the point round the circle of reference.

Motion from right to left is negative and from left to right positive. Displacement to the right of the center is positive; to the left negative. The *Phase* is the fraction of a period since the point last passed through the middle of its path in the positive direction.

21. ACCELERATION IN S. H. M. PROPORTIONAL TO DISPLACEMENT.

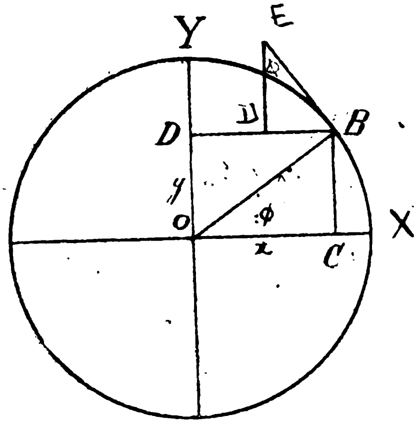


FIGURE 7.

Let the point be at B, Fig. 7, in the auxiliary circle. Its acceleration toward the center is $\frac{4\pi^2 r}{T^2}$. Resolve this acceleration parallel to O X and O Y. Denote the two component accelerations by f_x and f_y. Then

cos(90° - θ)

$$f_x = \frac{4\pi^2 r}{T^2} \cos \varphi ; \; f_y = \frac{4\pi^2 r}{T^2} \sin \varphi .$$

Call the displacement O C of the point x and the radius r; then $\cos \varphi = \frac{x}{r}$ and $\sin \varphi = \frac{y}{r}$.

$$\text{Hence } f_x = \frac{4\pi^2 r}{T^2} \times \frac{x}{r} = \frac{4\pi^2 x}{T^2};$$

$$f_y = \frac{4\pi^2 r}{T^2} \times \frac{y}{r} = \frac{4\pi^2 y}{T^2}.$$

The first varies as x and the second as y, or the two components of the centripetal acceleration are proportional to the displacement of the point from the two rectangular diameters. Hence uniform circular motion may be resolved into two S. H. M.'s at right angles, of the same period and amplitude, and differing in phase by a quarter of a period.

22. VELOCITY AT ANY POINT. (A. & B., 23.)

To find the velocity of the S. H. M. at any point C, resolve the uniform velocity V in the auxiliary circle into two components parallel to the X and Y axes. The component of V parallel to the X axis is —V sin φ; or since $V = \frac{2\pi r}{T}$,

$$V_x = -\frac{2\pi r}{T}\sin\varphi,$$

$$\text{and } V_y = \frac{2\pi r}{T}\cos\varphi.$$

When $\varphi = 90^\circ$, $\sin\varphi = 1$ and $\cos\varphi = 0$. When one component of the velocity is a maximum the other is a minimum and the sum of their squares is V^2. The velocity of S. H. M. at its maximum is the same as the uniform velocity in the circle of reference.

23. S. H. M's ISOCHRONOUS. (D., 82).

Since accelerations in S. H. M. are proportional to displacement, the *period* of the S. H. M. is independent of the amplitude of oscillation.

Let $\frac{2\pi}{T} = \omega$, angular velocity. Then $T = \frac{2\pi}{\omega}$. This is independent of amplitude if ω is constant.

$$\text{Acceleration is } \frac{4\pi^2}{T^2}x, \text{ or } \left(\frac{2\pi}{T}\right)^2 x, \text{ or } \omega^2 x.$$

Therefore $\omega = \sqrt{\frac{\text{acceleration}}{\text{displacement}}}$. The condition therefore that the period shall be independent of the amplitude is that $\frac{\text{acceleration}}{\text{displacement}}$ shall be a constant. This is true in S. H. M.

since acceleration is proportional to displacement; or on the other hand, since ω is a constant, because the motion of the point in the auxiliary circle is uniform.

24. GENERAL EQUATIONS OF S. H. M.

If x and y are displacements parallel to the two axes of reference, then

$$x = a \cos \theta;$$
$$y = b \sin \theta; \quad = [illegible]$$

in which a and b are amplitudes in the two rectangular directions. Let t be the time of describing angle θ. Then

$$2\pi : T :: \theta : t, \text{ or } \theta = \frac{2\pi t}{T}.$$

$$\text{Hence } x = a \cos \frac{2\pi t}{T};$$
$$y = b \sin \frac{2\pi t}{T}.$$

But both t and θ may be reckoned from some point at an angular distance of ε from X. This angle is called the epoch. Then

$$x = a \cos \left(\frac{2\pi t}{T} - \varepsilon\right);$$
$$y = b \sin \left(\frac{2\pi t}{T} - \varepsilon\right).$$

Since the number of complete oscillations n per second is the reciprocal of the time T, or $n = \frac{1}{T}$, we may write

$$x = a \cos (2\pi nt - \varepsilon);$$
$$y = b \sin (2\pi nt - \varepsilon).$$

$$\text{Also } V_x = \frac{2\pi a}{T} \sin (2\pi nt - \varepsilon);$$

$$f_x = \frac{4\pi^2 a}{T^2} \cos (2\pi nt - \varepsilon).$$

S. H. M. is therefore an oscillation in which the acceleration is proportional to the cosine of an angle which varies directly as the time.

25. COMPOSITION OF TWO SIMULTANEOUS CIRCULAR MOTIONS. (D., 99.)

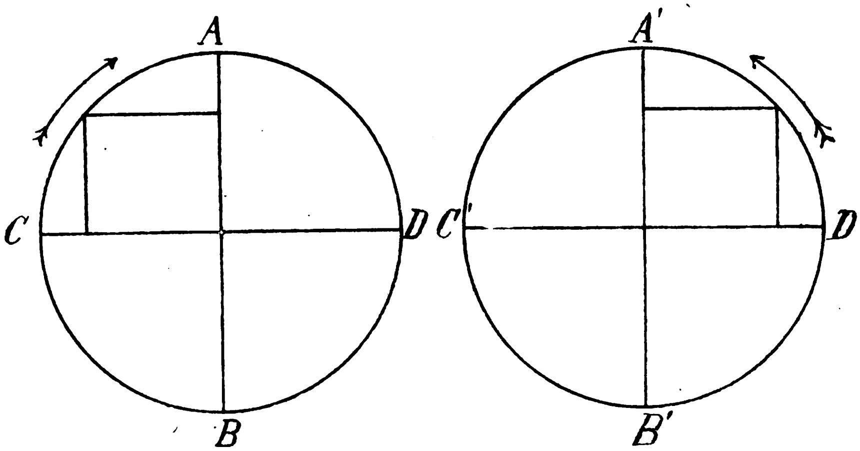


FIGURE 8.

Let the two motions, Fig. 8, be in opposite directions round circles of equal radii, and let them be of the same period. Resolve the two motions in rectangular directions parallel to A B and C D. Then the components parallel to C D are equal and oppositely directed and have a resultant of zero. Those parallel to A B are in the same direction, are both simple harmonic, and their resultant is a S. H. M. of the same period as the circular motions, and with amplitude double the radius of the circles. The resultant S. H. M. is in a plane of symmetry with respect to the two circles. If the periods of the two circular motions are not exactly equal, then the plane of symmetry revolves in the direction of the circular motion of shorter period, and makes a complete revolution while one circular motion gains a complete rotation on the other.

Conversely, S. H. M. may be considered as compounded of two uniform circular motions of like period and of radius equal to half the amplitude of the S. H. M.

KINETICS.

26. MOMENTUM. (A. & B., 17.)

Momentum is the product of mass and velocity. The mass of a body is the quantity of matter it contains. Hence

$$\text{Mom.} = mv.$$

Momentum has often been defined as "quantity of motion."

27. FORCE. (A. & B., 28.)

Force is said to act on a body whenever any change occurs or tends to occur in its *motion*. Force being known by the change it produces in the motion of a body, it is measured by the acceleration it imparts to unit mass. The formula for total force is

$$F = m\frac{v - v_o}{t} = m\,f.$$

Distinguish between the *intensity* of a force and the *total magnitude* of a force. The intensity of a force is the force acting on unit mass and is measured by the acceleration it produces.

The total magnitude of a force is the product of the mass and acceleration, or the total force acting on the entire mass.

A so-called *impulsive* force acts only for a very short period, while a continued force acts for a longer period. The only difference is one of time.

Impulse is the product of the force and the time during which it acts.

The *practical unit* of force is the *dyne*. It is that force which, acting on a gramme mass for one second, will produce a velocity of one cm. per second. In other words it will produce unit acceleration in unit time. Since force is measured by acceleration, we may substitute force for acceleration in the preceding propositions.

28. FIELD OF FORCE.

The region within which force acts is called a *field of force.* A *line of force* is such a line that its tangent at any point is the direction of resultant force at the point.

29. NEWTON'S LAWS OF MOTION. (A. & B., 30; M. & M., 55; T. & T., 65.)

Law 1.—Every body continues in its state of rest or of uniform motion in a right line, except in so far as it may be compelled by impressed force to change that state.

Law 2.—Change of motion is proportional to force applied and takes place in the direction of the straight line in which the force acts.

Law 3.—To every action there is always an equal and contrary reaction; or, the mutual actions of any two bodies are always equal and oppositely directed.

By "*change of motion*" Newton meant what is now called *change of momentum;* and by "*impressed force,*" *impulse.*

30. DISCUSSION OF THE LAWS OF MOTION. (A. & B., 31.)

The first law expresses the fact that motion as well as rest is a natural state of matter; that moving bodies when entirely free to move, proceed uniformly along straight lines; and that any deviation from uniform rectilinear motion is due to the action of force.

The second law shows the relation between the change of momentum and the impulse when a change of motion takes place. It forms the basis for the measurement of force and identifies the direction of the force with the direction of change of motion.

All action between bodies or between the parts of a system of bodies is of the nature of a *stress.* A stress is always a two-sided phenomenon. Considered only with respect to one portion of a system of bodies it is called *action;* with respect to the remainder of the system it is called *reaction.* The third law states that these two phases of a stress are always equal and oppositely directed.

31. Work. (A. & B., 34; M. & M., 101; D., 39; Stewart's Conservation of Energy.)

When force produces motion it is said to do work, and the work done is measured by the product of the force and the space through which motion is produced in the direction of the force. Thus

$$W = F s = m f s.$$

from Maxwell

"Work is the act of producing a change of configuration in a system in opposition to a force which resists that change." The unit of work in the C. G. S. system is the *erg*, or the work done by a dyne through one cm.

The megalerg = 10^6 ergs.
" ergten = 10^{10} "
" ~~watt~~ joule = 10^7 " ~~per sec.~~

The ~~*watt*~~ joule is properly a unit of *power*, which is the *rate of doing work.*

1 H. P. equals 33,000 ft. ℔s. per minute or 550 ft. ℔s. per second.

1 H. P. equals 746 × 10^7 ergs per second or 746 watts ~~per second~~.

Gravity gives to a gramme a velocity of 981 cm. per second. It is therefore equal to 981 dynes. To lift a gramme through a vertical height of 1 cm. requires the expenditure of 981 ergs of work.

The watt = 10^7 ergs per sec.

32. Energy. (Same references.)

"Energy is the capacity of doing work." A body, in consequence of its motion or its position with respect to other bodies, may have the capacity of doing work. This capacity is called energy. Energy is *kinetic* or *potential* according as it is due to *motion* on the one hand or *relative position* on the other. Kinetic energy is also called moving energy, and energy of motion. Potential energy means the same as energy of position, static energy, energy of stress. For illustrations consult lectures.

33. KINETIC ENERGY IN TERMS OF MASS AND VELOCITY.

For uniformly accelerated motion

$$v^2 = 2\,f\,s.$$

Hence $\frac{1}{2}\,m\,v^2 = m\,f\,s.$

But $m\,f\,s$ is the work done upon the body in producing motion. It is therefore the measure of the energy stored up in the body and equals $\frac{1}{2}\,m\,v^2$, where v is the final velocity.

Or otherwise: If F is the force acting on mass m during time t, then the impulse is Ft.

By the second law of motion impulse equals the change of momentum. If the initial and final velocities are v_0 and v, the change in momentum is $m(v-v_0)$. Hence,

$$Ft = m\,(v-v_0) \;\ldots\ldots\ldots\ldots\ldots\ldots (a).$$

The mean velocity is $\frac{1}{2}\,(v+v_0)$. It is also $\frac{s}{t}$. Hence

$$\frac{s}{t} = \frac{1}{2}\,(v+v_0) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (b).$$

Multiply (a) and (b) together and

$$Fs = \frac{1}{2}\,m\,v^2 - \frac{1}{2}\,m\,v_0^2.$$

Here Fs is the work done or the kinetic energy stored up. The increase in kinetic energy is therefore one-half the product of the mass and the difference of the squares of the initial and final velocities. If the initial velocity is zero, $Fs = \frac{1}{2}\,mv^2$ as before.

34. PRINCIPLE OF CONSERVATION OF ENERGY.

Consult Balfour Stewart's "Conservation of Energy," Maxwell's "Matter and Motion,' and refer to lectures.

35. TRANSFORMATIONS OF ENERGY.

(Same references as in 34.)

36. AVAILABILITY OF ENERGY.

(Refer to lectures.)

37. THE MOMENT OF A FORCE.

The moment of a force with respect to a point is the measure of the tendency of the force to produce rotation about the point. "The moment of any physical agency is the numerical measure of its importance." The moment of a force is numerically the product of the force and the perpendicular from the point upon the direction of the force. Thus, if O, Fig. 9, is the point and A B the force, the moment is

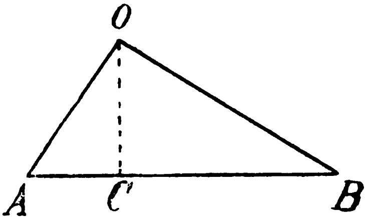


FIGURE 9.

$$F \times OC = AB \times OC.$$

This is double the area of the triangle A B O. O C is called the *lever arm*, and O the *origin*. If the moment of a force producing rotation in one direction is positive, that producing rotation in the other direction is negative.

38. THE MOMENT OF THE RESULTANT EQUALS THE ALGEBRAIC SUM OF THE MOMENTS OF THE COMPONENTS.

Let the origin O, Fig. 10, fall outside the angle included between the two forces P and Q. Complete the parallelogram and draw the diagonal A D for the resultant R. Complete figure as shown. Then p, q, r are the lever arms of the three forces, all producing rotation in the same direction. From similar triangles O A B^1, A B E,

$$AE : p :: AB : AO \quad \therefore AE = \frac{AB \times p}{AO}.$$

Similarly A F : q : : A C : A O $\quad \therefore A F = \frac{A C \times q}{A O}$;

and A G : r : : A D : A O $\quad \therefore A G = \frac{A D \times r}{A O}$.

[illegible]

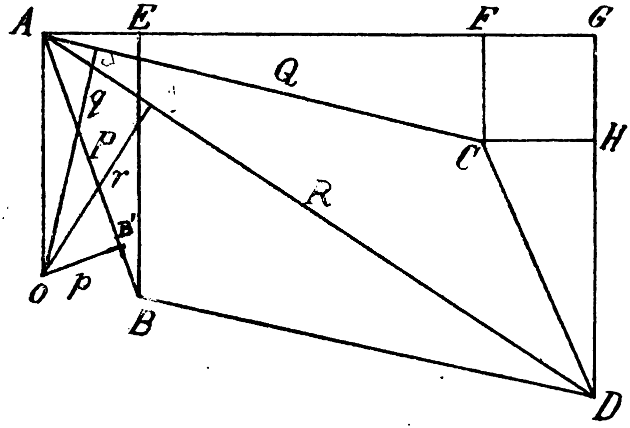


FIGURE 10.

But A G = A F + F G = A F + A E.
Therefore A D × r = A C × q + A B × p,
or R . r = P . p + Q . q .... Q. E. D.

If O lay within the angle B A C, then the moment of one force would be positive and that of the other negative.

Since the proposition is true for two forces and their resultant, it is true for this resultant and a third force, and so on indefinitely. If the origin O is in the line of the resultant or this line produced, then the moment of the resultant is zero; and the sum of the moments producing rotation in one direction must equal the sum of those in the other.

39. PARALLEL FORCES.

If C, Fig. 11 and 12, be the origin of moments then

P × D C = Q × E C. C D = lever arm

Whence P : Q : : E C : D C. [illegible]

But E C : D C : : B C : A C.

$\therefore$ P : Q : : B C : A C,

or the line joining the points of application of the two forces is divided by the resultant into parts inversely proportional to the forces. Also

$P : P + Q :: BC : BC + AC$,

or $P : R :: BC : AB$, Fig. 11;

and

$P : P - Q :: BC : BC - AC$,

or

$P : R :: BC : AB$, Fig. 12.

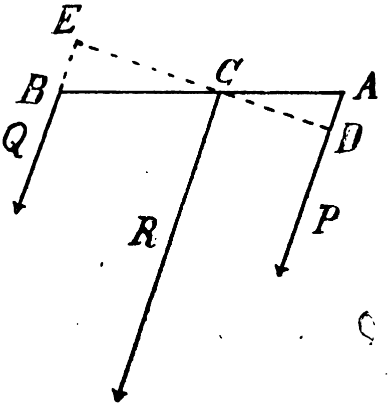


FIGURE 11.

Wherefore $P : Q : R :: BC : AC : AB$.

Each force is proportional to that part of the line joining their points of application, which is included between the other two.

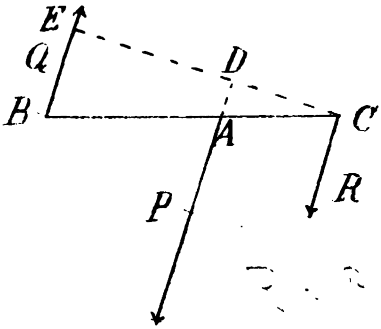


FIGURE 12.

The value of R has thus far been assumed equal to P + Q and P − Q respectively. To demonstrate, take E as origin. Then

$$P \times DE = R \times CE.$$

Whence $P : R :: CE : DE$.

But from the first proportion

$$P : P \pm Q :: CE : DE.$$

Therefore $R = P \pm Q$.

40. COUPLES.

In Fig. 12 P : P − Q : : B C : A B,

or $\frac{P}{R} = \frac{B\,C}{A\,B}$. If now P = Q, then the first member of the equation is infinite. Hence B C is infinite since A B is not zero. Such a system of two equal and parallel forces, oppositely directed, constitutes a *couple*. The moment of a couple is the product of either force and the perpendicular distance between the lines representing the forces.

41. MECHANICAL POWERS. THE LEVER. (A. & B., 47.)

If two weights, P and W, Fig. 13, be attached to a lever without weight at A and B, and the lever be supported at the point of application of the resultant of P and W by a cord passing over a pulley, and carrying a counterweight P + W, then there will be equilibrium. Hence, either A, B, or C may be conceived as fixed and the equilibrium will still subsist.

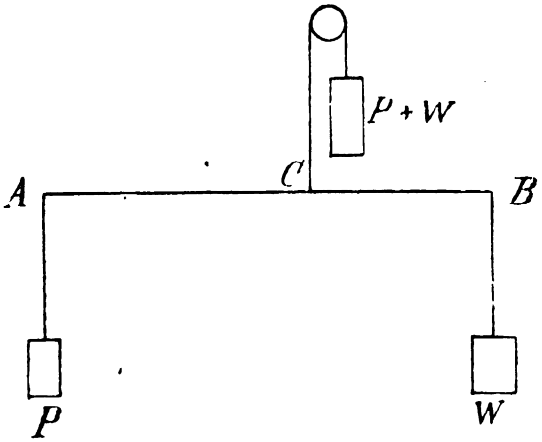


FIGURE 13.

Thus if C be fixed it becomes the fulcrum, and by the principle of moments P : W : : B C : A C.

This constitutes a lever of the first order. If A be fixed it is the fulcrum, and

$$W : P + W :: A\,C : A\,B,$$

or power and weight are inversely as their acting distances. If W is the power the lever belongs to the so-called second order.

But if P + W is the power and W the weight, the lever is of the third order.

The same relation of power and weight, expressed above, applies to the several orders alike.

42. PRINCIPLE OF LEVER BY THEORY OF WORK.

Suppose the system represented by the figure to suffer a small displacement, so that A and B, Fig. 14, take the positions A′ and B′. Then the positive work done on one side must equal the negative work on the other. If the displacement is small, the arcs A A′ and B B′ are the distances worked over. Hence

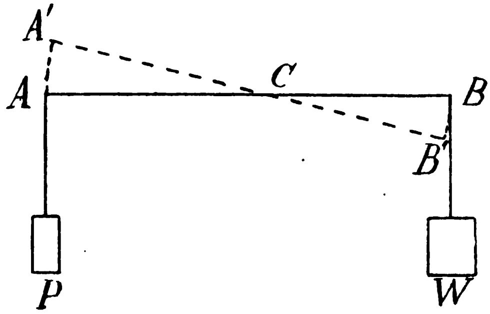


FIGURE 14.

$$P \times A\,A' = W \times B\,B',$$

$$\text{or } P : W :: B\,B' : A\,A'.$$

But similar arcs are to each as their radii.

Hence $P : W :: B\,C : A\,C$,

or power and weight are inversely as their lever arms or acting distances.

43. THE INCLINED PLANE.

Since accelerations are the measure of forces, the relation already found to exist between the acceleration of gravity and acceleration down an inclined plane also applies to power and weight, when the power is applied parallel to the plane.

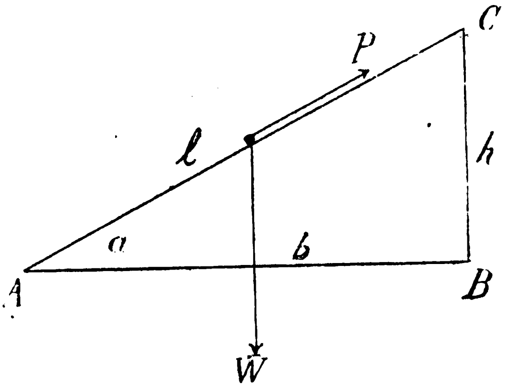


FIGURE 15.

Since $g' = g \sin \alpha$, $P = W \sin \alpha$;

or $P = W \frac{h}{l}$ and $P : W :: h : l$.

We obtain the same result by the principle of work. For the work done by P is equal to work done against gravity on W. While W is lifted through a vertical height *h*, P works over the length of the plane *l*.

Hence $P \cdot l = W \cdot h$, or
$P : W :: h : l$.

If the power is applied parallel to the base of the plane, then while the weight is carried from A to C, the power, acting parallel to A B, produces a displacement in its own direction equal to the base A B.

Hence $P \cdot b = W \cdot h$,
or $P : W :: h : b$.

44. SENSIBILITY OF THE BALANCE. (Desch., 36; A. & B., 78.)

Let the three points A, B, C, Fig. 16, be in the same straight line. A and B are the knife-edges supporting the scale pans, and C is the knife-edge supporting the beam. Let a weight P be applied on one side and P + p on the other. Then, upon producing a small displacement, the two lever arms remain equal to each other, P and P have equal moments and counterbalance each other, and it is necessary to take into account only the small weight *p* and the weight of the beam, which we may call *w*. The sensibility of the balance will depend upon the angular displacement of the beam, with a given small difference in weights *p*.

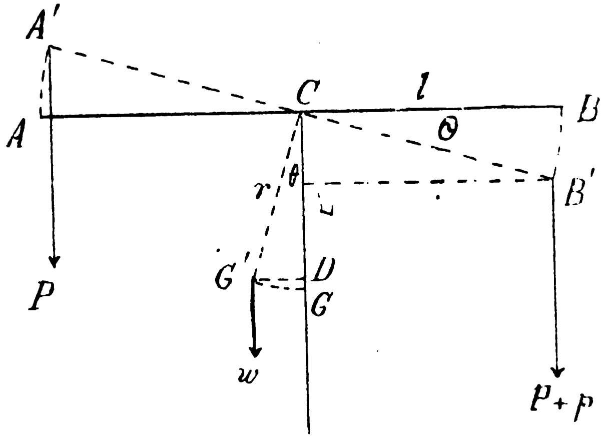


FIGURE 16.

When the beam assumes its position of equilibrium A′ B′, the moment of *p* on one side equals the moment of *w* on the other, C being the origin. Then

$$p \times B'L = w \times G'D.$$

G is the center of gravity of the beam; let C G be called *r*. If half the beam is represented by *l*, then

$$p \times l \cos \theta = w \times r \sin \theta.$$

Therefore $\frac{\sin\theta}{\cos\theta} = \tan\theta = \frac{p \,.\, l}{w \,.\, r}$.

The tangent of the angular displacement, or the sensibility if the angle is small, varies directly as the difference of weights and the length of the beam, and inversely as the weight of the beam and the distance between its center of gravity and the axis of suspension.

If the three points, A, B, C, are not in the same straight line the sensibility changes with change of load. Flexure of the beam has the same effect.

45. Double Weighing.

If the arms of the balance are of unequal length, the true weight of a body can still be found by the process of double weighing. The same procedure enables one to determine the ratio of the two arms of the balance. Let l and l' be the lengths of the two arms A C and C B. Let the body of true weight w be first suspended from A and counterbalanced by weight w' in the other scale pan. Then let it be placed in pan B and counterbalanced by weight w''.

Then for the first operation, $l\,w = l'\,w'$.

For the second, $l'\,w = l\,w''$.

Whence $l l'\,w^2 = l l'\,w'\,w''$,

or $w = \sqrt{w'\,w''}$.

Also $l^2\,w\,w'' = l'^2\,w\,w'$,

or $\frac{l}{l'} = \sqrt{\frac{w'}{w''}}$.

46. Center of Inertia. (A. & B., 46.)

If all the particles of a body or system of bodies be supposed collected at a point so situated that the same acceleration would be produced by applying to this point the resultant of all the separate forces acting on the several particles as by the conjoined action of all the separate forces, then this point is called the *center of inertia.*. It is commonly used synonomously with the expression *center of gravity.*

When a body consists of a finite number of parts, the masses and centers of inertia of which are known, then the distance of the center of inertia of the whole body from any plane may be found by taking the sum of the products of the several masses by their respective distances from the plane and dividing by the sum of the masses.

The center of inertia is thus the average distance of all the masses from the plane. Expressed in the form of a general formula,

$$X = \frac{\Sigma m x}{\Sigma m}.$$

It is demonstrated as follows: Let M M′, Figure 17, be the plane of reference. Let the particles m_1, m_2, be situated at A and B, distant respectively x_1 and x_2 from the plane. Let C be the center of inertia of the two particles. Through C draw D E parallel to F G. Then

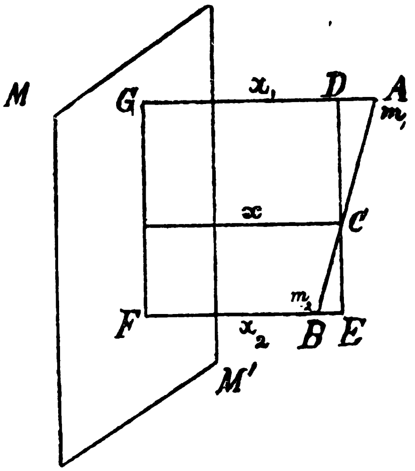


FIGURE 17.

$$m_1 : m_2 :: B\ C : A\ C :: B\ E : A\ D.$$

Therefore $m_1 \times A\ D = m_2 \times B\ E.$

But $A\ D = x_1 - x$ and $B\ E = x - x_2.$

Hence $m_1 (x_1 - x) = m_2 (x - x_2).$

$$\therefore \quad x (m_1 + m_2) = m_1 x_1 + m_2 x_2,$$

or generally $X \Sigma m = \Sigma m x.$

Whence $X = \frac{\Sigma m x}{\Sigma m}$.

If the plane passes through C then X is zero, and consequently

$$\Sigma m x = 0.$$

47. Center of Inertia of a Triangle.

Let the triangle be of uniform thickness and density. Divide it into a large number *n* of small areas by equidistant lines parallel to the base; from the apex C, Fig. 18, draw a line to the middle point of the base. The C.'s of I. of all the areas lie on this line, and therefore the C. of I. of the entire triangle lies on the same line. Let it be at G. Then find the distance of G from a line passing through C and perpendicular to C D. The areas of the small divisions, and therefore their masses, are as 1, 2, 3,.............. n.

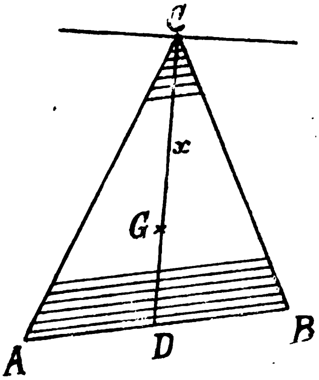


Figure 18.

The masses are equal to these numbers multiplied by a constant.

The products of masses and distances are as

$$1^2, 2^2, 3^2, \ldots\ldots\ldots\ldots n^2.$$

$$\text{Hence C G or } X = \frac{\Sigma m x}{\Sigma m} = \frac{1^2 + 2^2 + 3^2 + \ldots . n^2}{1 + 2 + 3 + \ldots . n}.$$

Applying formula $s = \frac{n^{m+1}}{m+1}$ (Art. 10, Note),

$$X = \frac{n^3}{3} \div \frac{n^2}{2} = \frac{2}{3} n, \text{ or } \frac{2}{3} \text{ C D}.$$

48. Center of Inertia of a Pyramid.

The pyramid may have any base. Conceive it divided into sections by a large number n of equidistant planes parallel to the base. Draw Cg, Fig. 19, from the apex to the C. of I. of the base. This line passes through the C. of I. of all the elementary areas, and hence through the C. of I. of the pyramid. Suppose it to be at G. To find C G suppose a plane passed through C perpendicular to Cg. Then the masses of the small divisions of the pyramid are as $1^2, 2^2, 3^2 \ldots\ldots n^2$.

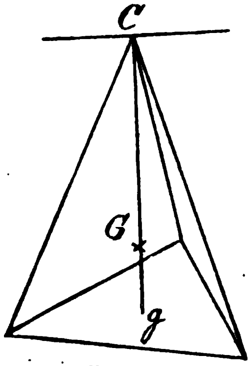


FIGURE 19.

The products are as

$$1^3, 2^3, 3^3 \ldots\ldots n^3.$$

$$\text{Hence, C G or X} = \frac{\Sigma\, m\, x}{\Sigma\, m} = \frac{n^4}{4} \div \frac{n^3}{3} = \frac{3}{4}\,n, \text{ or } \frac{3}{4}\,\text{Cg}.$$

49. MOMENT OF INERTIA. (A. & B., 58.)

The moment of inertia of a body about an axis is the sum of the products of the masses composing the body and the squares of their distances from the axis. It measures the importance of the body's inertia with respect to rotation, and is proportional to the kinetic energy of the body revolving about the axis with a constant angular velocity.

Angular velocity is the rate at which the angle θ between any movable radius and some fixed radius is described; or

$$\omega = \frac{\theta}{t}, \text{ in which } \omega \text{ is the angular velocity.}$$

Angular acceleration is the rate of change of angular velocity; or

$$\varphi = \frac{\omega - \omega_0}{t}, \text{ in which } \varphi \text{ is angular acceleration.}$$

If a particle *m* is at a distance *r* from the axis of rotation and rotates with angular velocity ω, its linear velocity is $r\ \omega$. Hence its kinetic energy is $\frac{1}{2}\ m\ r^2\ \omega^2$, and the kinetic energy of the whole body is $\frac{1}{2}\ \omega^2\ \Sigma\ m\ r^2$, since ω is the same for all parts of the body. $\Sigma\ m\ r^2$ is called the moment of inertia. The work done upon a body to give it the angular velocity ω is represented by its kinetic energy of rotation, and it is therefore proportional to the moment of inertia of the body.

If we can find a distance *k* such that

$$\tfrac{1}{2}\ k^2\ \omega^2\ \Sigma\ m = \tfrac{1}{2}\ \omega^2\ \Sigma\ m\ r^2,$$

then *k* is called the radius of gyration.

The moment of inertia may also be defined as the moment of the couple required to produce unit angular acceleration. If φ is angular acceleration, then the linear acceleration of a particle *m* distant *r* from the axis of rotation is $r\ \varphi$, and the total force producing this acceleration is $m\ r\ \varphi$; the moment of this force about the axis is $m\ r^2\ \varphi$. The total moment of the entire force producing the rotation is

$\Sigma\ m\ r^2\ \varphi = \varphi\ \Sigma\ m\ r^2$, since φ is the same for all parts of the body. If now φ. is unity, then $\Sigma\ m\ r^2$ is the moment of the force or couple required to produce unit angular acceleration.

50. Moment of Inertia About a Parallel Axis.

Let G, Fig. 20, be the C. of I. of the body, and let the parallel axis pass through A, the distance between the two being *d*. Let a small mass *m* be situated at B. Then in the triangle A B G,

$$\rho^2 = r^2 + d^2 - 2\ r\ d \cos A\ G\ B.$$

But $\quad r \cos A\ G\ B = x.$ Hence

$$\rho^2 = r^2 + d^2 - 2\ d\ x.\ \text{ Then}$$

$m \rho^2 = m r^2 + m d^2 - 2 m d x$; and for the entire body,

$$\Sigma m \rho^2 = \Sigma m r^2 + \Sigma m d^2 - 2 d \Sigma m x.$$

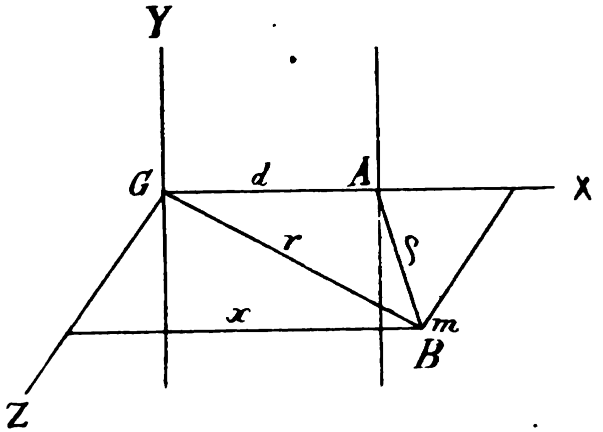


FIGURE 20.

The last term is zero (Art. 46).

If I_0 represent M. of I. round axis through C. of I. and I that round any parallel axis, then

$$I = I_0 + M d^2.$$

The M. of I. round any parallel axis is equal to the M. of I. round axis through C. of I. plus the product of the mass of the body and the square of the distance between the two axes.

51. THE IDEAL SIMPLE PENDULUM.

By an ideal simple pendulum is meant one in which the entire mass is collected at a point and suspended by a thread without weight. Let *m* be suspended from O, Fig. 21, by thread of length *l*. In the position of C the force of gravity *m g*, acting on the mass, may be resolved into two components, one in the direction of the string, ineffective in producing motion, and the other tangential, the effective component. The tangential component is m g sin θ. If the mass is unity

$$f = g \sin \theta.$$

If θ is small, sin θ equals θ, or

$$f = g\,\theta = g\,\frac{x}{l} = \frac{g}{l}\,x.$$

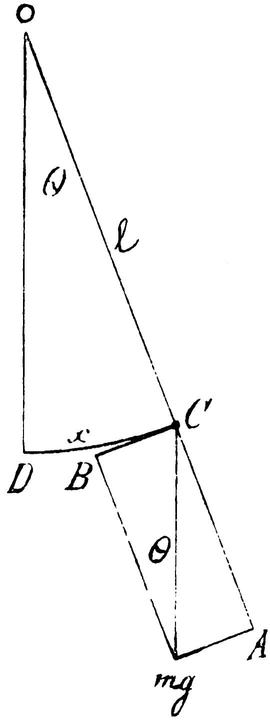


FIGURE 21.

The acceleration is proportional to x, the displacement of the pendulum from the point of rest. The motion is therefore simple harmonic within the limits of the approximation made that sin $\theta = \theta$.

In S. H. M. $f = \frac{4\,\pi^2}{T^2}x$. If we put the constant $\frac{4\,\pi^2}{T^2} = \mu$, then $\frac{2\,\pi}{T} = \sqrt{\mu}$ and $T = \frac{2\,\pi}{\sqrt{\mu}}$, a general formula for the period in S. H. M. But the constant in the case of the simple pendulum is $\frac{g}{l}$. This is therefore the value of μ for the simple pendulum. Substituting above and $T = 2\,\pi\sqrt{\frac{l}{g}}$. For a single vibration $T = \pi\sqrt{\frac{l}{g}}$.

The times of vibration of pendulums of different length are as the square roots of those lengths.

If T is one second, then

$$1 = \pi\sqrt{\frac{l}{g}}, \text{ or}$$

$$g = \pi^2\,l.$$

Since $\pi^2 = 9.869$ and $g = 981$, $l = \frac{981}{9.869} = 99.4$ cm. If gravity equaled 986.9 the length of the seconds pendulum would be 100 cms. or 1 meter.

52. THE COMPOUND OR PHYSICAL PENDULUM.

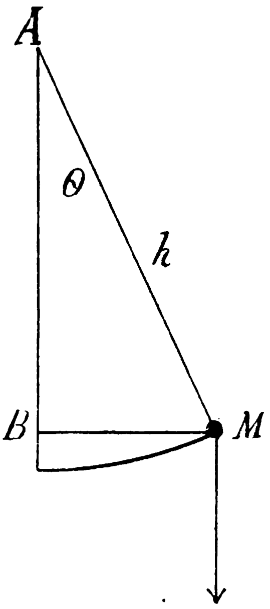


FIGURE 22.

Let the mass of the heavy pendulum, including its suspension, be represented by M, Fig. 22, and let the C. of I. of this mass be at a distance h from A, the axis of suspension. Then the moment of the force producing rotation is

$$M\,g\,h \sin \theta.$$

But the moment of this force is also $\varphi\,\Sigma\,m\,r^2$ (Art. 49). Hence

$$\varphi\,\Sigma\,m\,r^2 = M\,g\,h \sin \theta,$$

$$\text{or } \frac{\Sigma\,m\,r^2}{M\,h} = \frac{g \sin \theta}{\varphi}.$$

This may be considered as the general formula. If now the entire mass M be conceived as collected at a point, at a distance l from A, so situated that its time of vibration as a simple pendulum is the same as that of the compound physical pendulum, then substituting in the general formula above,

$$\frac{\Sigma\,m\,r^2}{M\,h} = \frac{\Sigma\,m\,l^2}{M\,l} = \frac{M\,l^2}{M\,l} = l.$$

The length l is called the length of the equivalent simple pendulum. It is found by dividing the moment of inertia of the pendulum by the product of its mass and the distance between the axis of suspension and the center of mass.

Substitute this value of *l* in the formula for period in the last section and

$$T = \pi \sqrt{\frac{\Sigma m r^2}{M g h}}.$$

The numerator of the radical part of the expression is the *moment of inertia* of the pendulum, and the denominator is its *statical moment*.

53. The Reversibility of the Pendulum.

Let A, Eig. 23, be the axis of suspension and O the *center of oscillation*, distant *l* from A. Applying the principle of moment of inertia about a parallel axis,

$$l = \frac{\Sigma m r^2}{M h} = \frac{I_0 + M h^2}{M h}$$

for axis through A. If now the pendulum be suspended from axis through O and the length of the equivalent simple pendulum be *l'*, then

$$l' = \frac{\Sigma m r^2}{M (l - h)} = \frac{I_0 + M (l - h)^2}{M (l - h)}$$

for axis through O.

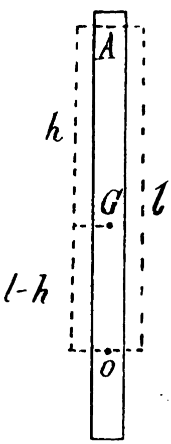


FIGURE 23.

From the first equation

$$l M h = I_0 + M h^2, \text{ or } I_0 = M h (l - h).$$

Substituting in the second equation,

$$l' = \frac{M h (l - h) + M (l - h)^2}{M (l - h)} = \frac{h + (l - h)}{1} = l,$$

or the length of the equivalent simple pendulum is the same whether the axis of suspension be through A or O; and, therefore, the time of vibration is the same. This principle furnishes a method, made use of in Kater's pendulum, of determining the length of the equivalent simple pendulum. It is the distance between two axes through A and O, so related that the pendulum vibrates in the same time about the two.

MECHANICS OF FLUIDS.

The term fluid applies both to liquids and gases. A *perfect fluid* would offer no resistance to shearing stress. The fluid most nearly perfect offers some resistance to shearing stress due to its viscosity. All such fluids at rest can be treated as if they were perfect in determining conditions of equilibrium. Hence

54. No Statical Friction in Fluids.

By statical friction is meant the friction existing between bodies at rest. Statical friction acts tangential to the surfaces in contact. The absence of statical friction in fluids means that all fluid pressure is normal to the surface pressed upon. When fluids are in motion a tangential force of friction exists. Hence the middle of a stream moves faster than the sides. A glacier is known to move down its bed in the same manner. The tangential friction in fluids is due to a property, called *viscosity*, possessed by all fluids in different degrees.

55. Pascal's Law.

In a fluid at rest, not acted on by any external forces except the pressure of the containing vessel, the pressure at every point and in every direction is the same. For conceive a small cube of the fluid to assume the solid state without change of volume. Then the pressure on each face of this cube is the same, whichever way it may be turned. If it were not so motion would ensue because there would be a resultant. But by hypothesis the fluid is at rest. Hence the pressure in all directions is the same.

56. Pressure the Same at all Points of a Horizontal Plane.

If external forces, such as gravity, act on the fluid, then Pascal's principle no longer holds good. For in order that the cube above considered shall be in equilibrium, the pressure on its lower surface must be greater than that on its upper surface by the weight of the cube itself. But the pressures at all points of a horizontal plane still remain the same. For consider a small cylinder of the fluid with its axis horizontal and its ends equal vertical planes. Then since it is at rest the horizontal pressures on these planes will be equal and in opposite directions. But these planes may be anywhere in the horizontal plane. Moreover, the diameter of the cylinder may become indefinitely small. The difference between the vertical forces will then be infinitesimal. Hence the forces at all points of any horizontal plane are equal.

57. THE FREE SURFACE OF A LIQUID HORIZONTAL.

Conceive a particle m at some point A of the free surface, which we may suppose is not horizontal. The vertical force acting on this particle is $m\ g$. Resolve it into a normal and a tangential component. The tangential component, being unresisted, will produce motion. But the liquid is at rest. Hence there is no tangential component. It follows that the free surface is horizontal because the tangential component disappears only when the surface is horizontal. Such a surface is an equipotential surface, and the lines of force due to gravity are everywhere normal to it.

58. PRESSURE VARIES DIRECTLY AS THE DEPTH.

The pressure at any point in an incompressible fluid is proportional to its depth beneath the surface. For consider two vertical cylinders of a liquid with their upper bases equal and in the surface of the liquid in equilibrium. Let the length of one be n times that of the other. Then one weighs n times as much as the other, and equilibrium can exist only when the upward pressure on the base of the longer one is n times as great as on the other; for the upward pressure is balanced by the weight of the column of liquid. That is, the upward pressures are proportional to the depth below the surface.

59. TRANSMISSION OF PRESSURE IN FLUIDS.

Since the difference in fluid pressure on any two horizontal planes is proportional to their difference of level and is independent of the absolute pressure on either plane, it follows that an increase of pressure on one horizontal plane must be transmitted to every other one in the liquid, since the differences of pressure are constant, depending only on the weight of liquid between any two planes considered. Hence pressure applied at one point of a confined liquid is transmitted without diminution to every other point. This is the principle of the hydraulic press.

60. TWO LIQUIDS IN COMMUNICATING TUBES.

Let two liquids of different density be placed in connecting tubes. Then their heights above the horizontal plane through their common surface of separation will be inversely as their densities. For let the height of one above this plane be h and the other h', and let their densities be d and d'. Then since the pressure on the horizontal plane of separation is everywhere the same, the pressure due to the two heights h and h' must be the same, or

$$h\,d\,g = h'\,d'\,g. \quad \text{Therefore}$$

$$h : h' :: d' : d,$$

or the heights are inversely as the densities.

61. BACK PRESSURE ON A DISCHARGING VESSEL.

(Referred to lectures.)

62. TOTAL PRESSURE ON ANY IMMERSED SURFACE.

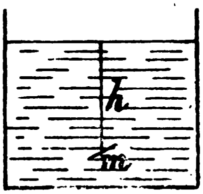


FIGURE 24.

Let *m*, Fig. 24, be any small immersed surface, the distance of which from the free surface of the liquid is *h*. Since it is small, the pressure on it is independent of its position relative to the horizontal. Let the total area, on which the pressure is to be found, be

$$A = m_1 + m_2 + m_3 + \ldots\ldots\ldots\ldots\ldots$$

The pressure on m_1 is $m_1\ h_1\ d\ g$; on m_2, $m_2\ h_2\ d\ g$, etc. Then the total pressure on the area A is

$$P = (m_1\ h_1 + m_2\ h_2 + m_3\ h_3 + \ldots\ldots\ldots\ldots)\ d\ g.$$

But by article 46 the quantities within the parenthesis may be put equal to A H, where H is the depth of the center of inertia of the entire surface A, the plane of reference being the surface of the liquid.

Hence $\qquad P = A\ H\ d\ g.$

This expression is the weight of a column of liquid, the base of which is the immersed surface and the height the distance of its center of inertia below the free surface of the liquid.

63. CENTER OF PRESSURE.

The center of pressure on any immersed surface is the point of application of the resultant of all the elementary hydrostatic pressures. For a plane area these elementary pressures are all parallel, and the problem consists in finding the resultant of a system of parallel forces. Thus to find the center of pressure on a rectangle with one side in the liquid surface, divide the rectangle, Fig. 25, into a very large number *n* of equal areas by lines parallel to the surface. Since the areas are equal, the pressures on them are simply proportional to the depth and the center of each is on the line O G. Let the surface of the liquid be the plane of reference; then applying the principle

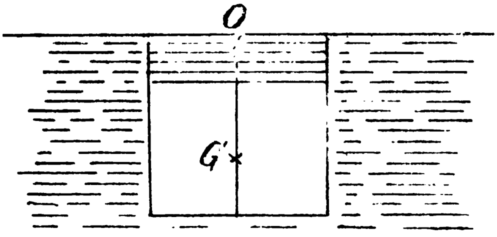


Figure 25.

of article 46, the pressures may be represented by

1, 2, 3, 4, ... n;

Moments

the products by $1^2, 2^2, 3^2, 4^2$ .... n^2.

Hence $O G = X = \frac{\Sigma m x}{\Sigma m} = \frac{n^3}{3} \div \frac{n^2}{2} = \frac{2}{3} n$, or $\frac{2}{3}$ the depth of the rectangle.

64. Center of Pressure on an Immersed Triangle.

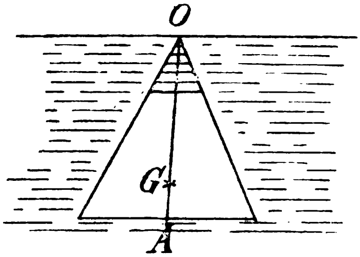


Figure 26.

Let the apex be at the surface and the base horizontal. Divide the triangle, Fig. 26, into a large number of sections by equidistant lines parallel to the base. Then the areas are

1, 2, 3, 4, ... n.

Pressures are as

$1^2, 2^2, 3^2, 4^2$ .... n^2.

Products are as $1^3, 2^3, 3^3, 4^3$ .... n^3.

$O G = X = \frac{\Sigma m x}{\Sigma m} = \frac{n^4}{4} \div \frac{n^3}{3} = \frac{3}{4}$ n, or $\frac{3}{4}$ the depth from the apex to the middle point of the base.

65. Principle of Archimedes.

Every body immersed in a liquid is subjected to an upward pressure equal to the weight of the liquid displaced. This follows if we conceive the body replaced by the liquid; for the upward pressure on the liquid must equal its own weight for equilibrium. The upward pressure on the immersed body is the same as on the replacing liquid. The same principle applies to all *fluids*.

The *center of buoyancy* is the name given to the center of gravity of the displaced liquid. The upward pressure acts vertically through this point.

66. DENSITY AND SPECIFIC GRAVITY.

Density is the mass per unit volume.

Whence $$M = V\,D, \text{ or } D = \frac{M}{V}.$$

By specific gravity is meant the ratio of the density of the body to that of another body taken as a standard. If s denotes the specific gravity of a body, D its density, and d the density of the standard, then

$$D = s\,d \text{ and } M = V\,s\,d.$$

If the density of the standard is unity, then densities and specific gravities are numerically equal. Thus, in the metric system, the mass of 1 c. c. of water is 1 gramme, or the density of water is unity under standard conditions. The density and specific gravity of bodies in this system are therefore equal. If, however, the English gravitation system is employed, then the density of water is about 62.4; since a cubic foot of distilled water at 4° C. contains a mass of 62.4 lbs.

67. SPECIFIC GRAVITY OF SOLIDS.

a. Bodies heavier than water. The specific gravity of solids insoluble in water may be found by weighing the body first in air and then suspended in water. Its apparent loss of weight in water is the weight of the water displaced. Hence, dividing the weight in air by the loss in water gives the specific gravity of the body.

b. Bodies lighter than water. Employ a sinker sufficient to make the body sink in water. Counterbalance with the body in the scale pan and the sinker suspended from it and immersed in water. Then transfer the body from the scale pan to the sinker in water. The apparent loss of weight is the weight of the water displaced. Divide as before for specific gravity. Other methods given in lectures.

68. Density of Liquids Inferred from Loss of Weight.

The specific gravity of liquids may be determined by means of Archimedes' principle. A glass sinker is weighed in air, and then the loss of weight it sustains in water and in the liquid under examination is determined. Let these apparent losses be w and w'. Then its specific gravity is $\frac{w'}{w}$, since its apparent loss of weight in the two cases is the weight of the same volume of the two liquids. The proper temperature corrections must be made in all cases.

69. Hydrometers of Variable Immersion—General Theory.

The approximate density of liquids is usually obtained by the use of hydrometers. The graduation of hydrometers must be made empirically. Those of equidistant divisions have their constants determined as follows: Let v be be the volume immersed, in units of the divisions of the stem, when the hydrometer sinks to zero. Then if d and d_1 are the densities of two liquids, and n and n_1 the corresponding divisions to which the instrument sinks in them, the zero being near the top, then the volumes immersed in the two cases are $v - n$ and $v - n_1$.

Hence $$(v - n)\, d = (v - n_1)\, d_1, \text{ or}$$

$$v = \frac{nd - n_1 d_1}{d - d_1}.$$

If the one liquid is water and the hydrometer sinks in it to zero, then $n = 0$ and $d = 1$.

Hence $$v = \frac{n_1 d_1}{d_1 - 1}.$$

For any other liquid of density D, in which the hydrometer sinks to division N, the equation of equilibrium is

$$D(v - N) = v.$$

and $$D = \frac{v}{v - N}.$$

70. BAUME'S HYDROMETER.

The two liquids are water and salt water containing 15 % of salt of density 1.11383 at 17°.5 C. The division to which the hydrometer sinks in the salt water is marked 15.

Then $$v = \frac{15 \times 1.11383}{0.11383} = 146.78,$$

and $$D = \frac{146.78}{146.78 - N}.$$

This formula gives the density corresponding to any division N.

For liquids lighter than water the zero of the scale is near the bottom of the stem. It is placed where the instrument sinks in a 10 % salt solution, and the point to which it sinks in water is marked 10. A general formula is obtained as in the other case by reversing the signs of n and n_1.

71. FUNDAMENTAL PHENOMENA OF CAPILLARY ACTION.

(A. & B., 91; Desch., 124.)

Elevation or depression of liquids in small tubes or by the side of plates; change of elevation or depression with size of bore; variation with different liquids; diminution of elevation or depression with rise of temperature, &c.

72. LAW OF FORCE ASSUMED.

The attraction of gravitation between masses of matter is too small to account for capillary phenomena. They can be accounted for if we assume an additional attraction between the molecules. The total expression for the stress between two molecules *m* and *m′* thus becomes

$$F = \frac{m\,m'}{r^2} + m\,m'\,f(r).$$

The first term expresses the attraction of gravitation; the second expresses the molecular attraction giving rise to capillary phenomena. This function of r is very large for insensible distances, diminishes rapidly as r increases, and vanishes while r is a very small quantity. According to Quincke the range of molecular action, called ϵ, is 0.000005 cm. or $\frac{1}{500000}$ inch. Within this distance the first term of the above expression is negligible in comparison with the second.

73. SURFACE TENSION. (A. & B., 94; Tait's Properties of Matter, 227.)

Consider a liquid bounded by a plane surface *m n*, Fig. 27, and let *m′ n′* be a parallel plane at a distance ϵ below the surface. If we consider a plane passed through any

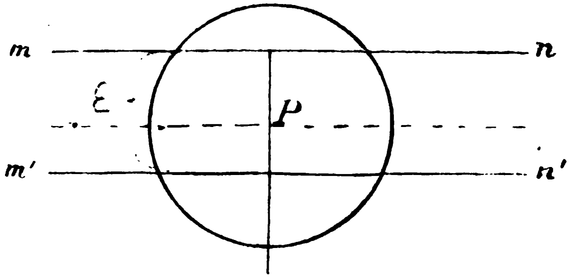


FIGURE 27.

point in the mass of the liquid below *m′ n′*, then the normal pressure on this plane due to molecular attraction is independent of the direction of the plane; for the number of molecules acting on the point is the same in every direction. If, however, the point is at P, nearer the surface than *m′ n′*, then about P as a center and with a radius ϵ, describe a sphere. Then the normal pressure upon the plane through P perpendicular to *m′ n′* is greater than upon the plane through P parallel to *m′ n′*. The pressure on the parallel plane varies from a maximum at *m′ n′* to zero at the surface. From this inequality of pressure there results a stress or tension producing an apparent contraction in the surface. For if the surface be increased by forcing molecules out along the plane through P normal to the surface, then work will be done upon the liquid against this molecular stress. But potential energy always tends to become a minimum. Hence the surface contracts to as small dimensions as the conditions allow, thus converting potential energy, due to surface tension, into some form of kinetic energy. Hence, every free liquid assumes the form of a sphere, in which the bounding surface is a minimum.

74. Energy and Surface Tension.

If we call the loss of potential energy, due to a diminution in the surface of one unit, the *surface energy* per unit surface, it can be shown that this is numerically equal to the surface tension across one unit width.

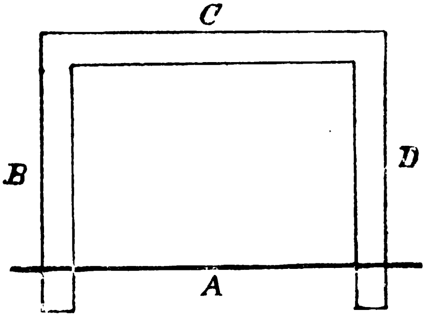


Figure 28.

Suppose a liquid film to be stretched on a frame A B

C D, Fig. 28, B C D being solid and A being a light rod moving freely along B and D. If the length of the rod is *a* and if it be drawn towards C over *b* units, then the diminution in surface is *a b* and in the surface energy is E *a b*, if E is the energy per unit surface.

Further, the work done by the tension in moving the rod is T *a b*, T being surface tension per unit width. Equating the two and canceling *a b*, E = T; or the energy per unit surface is equal to the surface tension per unit width.

75. Theory of Surface Tension Applied to Explanation of Elevation and Depression.

Let *h*, Fig. 29, be the elevation of the liquid in the tube above the level outside. If T is the surface tension for unit width, then the entire tension round the interior of the tube is 2 π r T. If the angle of contact of the film with the wall of the tube is θ, the vertical component of the tension, pulling the liquid up and the tube down, is 2 π r T cos θ. This is equilibrated by the liquid pressure of

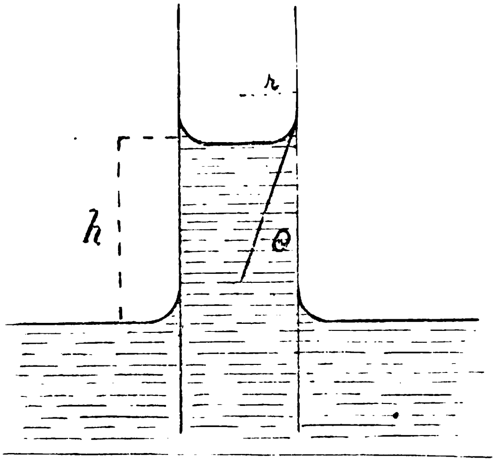


FIGURE 29.

the column downward, which is $\pi r^2 w h$, if w is the weight of unit volume of the liquid. Then

$$2 \pi r T \cos \theta = \pi r^2 w h.$$

Hence $h = \frac{2 T \cos \theta}{r w}$, or the elevation is inversely proportional to the diameter of the tube. If the angle of contact θ is more than 90°, then $\cos \theta$ is negative and the elevation becomes a depression. For water, the angle of contact with clean glass is supposed to be zero. Hence, for water and clean glass,

$$h = \frac{2 T}{r w}.$$

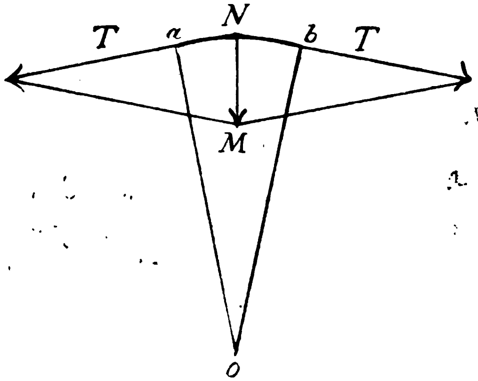
For two plates distant u from each other, the elevation is

$$h = \frac{2 T \cos \theta}{u w},$$

or the elevation is half as great as for a tube the diameter of which is u.

76. THE NORMAL PRESSURE ON A CURVED FILM.

Let $a\ b$ be a section of a cylindrical film. T is the



ab = S

FIGURE 30.

surface tension. Then is N M the normal pressure towards the concave side of the film. Let O be the center of curvation and R the radius. Then from similar Tri.

$$T : NM :: Oa : ab,$$

or

$$T : P :: R : S.$$

Whence $P = S\,\frac{T}{R}$,

in which S is the length of the strip $a\,b$ of unit width. If S is unity, then

$$P = \frac{T}{R}.$$

For two principal curvatures at right angles, of radius R and R_1,

$$P = T\left(\frac{1}{R} + \frac{1}{R_1}\right).$$

If the free film is plane, then both R and R_1 are infinite. If both sides are exposed to the air, while the film is still curved, then there can be no normal pressure, and

$$\frac{1}{R} + \frac{1}{R_1} = 0,$$

or the radii of curvature are equal and the centers are on opposite sides of the film.

77. Angles of Contact. (A. & B., 98).

Suppose three substances in contact to meet along a line, and let O, Fig. 31, be the point where this line cuts a

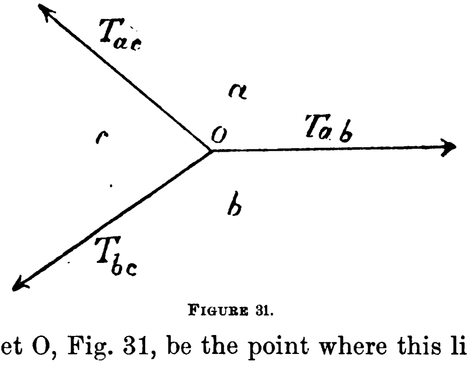


Figure 31.

plane at right angles to it. Then the tensions T_{ab}, T_{bc} and T_{ac} are in equilibrium, and the angles are constant for constant values of these tensions. If T_{ab} is greater than the sum of T_{bc} and T_{ac}, then the angle between T_{ac} and T_{bc} becomes zero and the fluid *c* spreads itself out in a thin sheet between *a* and *b*. This is the case with oil between air and water.

When two fluids *a* and *b* are in contact with a plane solid *c* and their surface of separation makes an angle θ with the solid, then the equation of equilibrium is

$$T_{ac} = T_{bc} + T_{ab} \cos \theta, \text{ and}$$

$$\cos \theta = \frac{T_{ac} - T_{bc}}{T_{ab}}.$$

If T_{ac} be greater than $T_{bc} + T_{ab}$ the equation gives an impossible value for cos θ, the angle becomes evanescent, and the fluid *b* spreads out and wets the entire solid surface. Thus a drop of water will spread out over a clean horizontal glass plate, while a drop of mercury will hold together, the edges making a constant angle with the plate.

78. AIR HAS WEIGHT.

Aristotle weighed a bladder inflated and collapsed. Galileo determined that water rose 32 ft. in the pumps of the Duke of Tuscany.

1 Liter of air weighs 1.293 gm.
1 cub. meter of air weighs 1.293 kilos.
1 " " " H " 0.0896 "

at 0° C and 760 mm. pressure.

79. ATMOSPHERIC PRESSURE.

The pressure of air on one square cm. of surface is

76~~0~~ × 13.596 = 1033.33 gms.,
or 1033.33 × 980 = 1,000,000 dynes.

One atmosphere is a pressure of 10^6 dynes.

80. THE BAROMETER.

Simple cistern barometer.
Fortin's "
The Aneroid "

81. HEIGHT OF THE HOMOGENEOUS ATMOSPHERE.

If the air were homogenous in density throughout its height, H in cms. could be determined in terms of pressure, density and gravity. For the pressure on one sq. cm. surface would be

$$P = H D g.$$

Whence $H = \frac{P}{Dg}$, the height of the homogeneous atmosphere. For the same temperature at different elevations P and D vary in the same ratio, and therefore H remains the same.

82. BOYLE'S LAW. (Mariotte's Law)

For the same temperature Boyle's law asserts that the volume of a gas is inversely as the pressure to which it is subjected; or

$$V : V' :: P' : P.$$

Whence $V P = V' P'$.

Since the density of a given mass of gas is inversely as the volume it occupies,

$$D : D' :: P : P'.$$

Densities are directly as pressures.

At high pressures Boyle's law is not exactly true, particularly for easily condensible gases.

If Boyle's law were exact then

$$\frac{V P}{V' P'} - 1 = 0.$$

Regnault found that for hydrogen the first term of this equation is not zero but has a negative value; for all other gases it has a positive value at ordinary temperatures. Hydrogen is less compressible than the law requires; other gases are more compressible.

83. THE AIR-PUMP.

Let V be the volume of the receiver and C that of the pump cylinder. Let *d* and d_1 be densities of air in the receiver before and after one stroke. For one stroke the volume V becomes V + C.

Hence $d : d_1 :: V + C : V$

or $$\frac{d_1}{d} = \frac{V}{V + C}.$$

$$\frac{d_2}{d_1} = \frac{V}{V + C}$$

After two strokes

$$\frac{d_2}{d} = \left(\frac{V}{V + C}\right)^2.$$

Generally

$$\frac{d_n}{d} = \left(\frac{V}{V + C}\right)^n = \left(\frac{1}{1 + \frac{C}{V}}\right)^n.$$

This expression becomes zero only when *n* is infinite.

84. UPWARD PRESSURE OF THE AIR.

Baroscope explained.

Lifting power of balloons.

1 cub. meter of air weighs 1.300 Kilos.
1 " " " H " 0.089 "
1 " " " coal-gas " 0.750 "

Hence to lift a balloon weighing 1500 kilos. requires

$$\frac{1500}{1.211} = 1239 \text{ cub. meters H};$$

$$\frac{1500}{.550} = 2727 \text{ " " coal-gas.}$$

85. EFFECT OF UPWARD PRESSURE ON WEIGHT OF BODIES.

Let x be the real mass of the body in grammes.
" w " " mass of the weights.
" d " " density of the body.
" δ " " " " " weights.
" a " " " " " air.

773

Then the volume of the body is $\frac{x}{d}$.

and " " " " weights is $\frac{w}{\delta}$.

Hence the masses of air displaced by the two are $\frac{a}{d}$ x and $\frac{a}{\delta}$ w respectively.

Therefore for equilibrium,

$$x - \frac{a}{d}x = w - \frac{a}{\delta}w.$$

$$\text{Whence } x = w\frac{1-\frac{a}{\delta}}{1-\frac{a}{d}} = w\left\{1 + a\left(\frac{1}{d} - \frac{1}{\delta}\right)\right\} \text{ nearly.}$$

If d is greater than δ the correction is negative; if d is less than δ, it is positive. When d is equal to δ the correction is zero.

86. TORRICELLI'S THEOREM FOR VELOCITY OF EFFLUX.

Let a small opening be made in the side of a vessel containing water; the depth of the orifice below the surface is h. Torricelli's law for the velocity of efflux is

$$v^2 = 2\,g\,h.$$

If we suppose a small mass m to issue from the orifice, an equal mass must have fallen a small distance a_1 to take its place; another equal mass has fallen a distance a_2 and so on through the series to the surface.

The total loss of potential energy is

$$m g a_1 + m g a_2 + m g a_3 + \ldots\ldots\ldots\ldots = m g h$$

in which h is the sum of a_1, a_2, a_3, etc.

Since what is lost in potential energy is gained in energy of motion, we may write,

$$\tfrac{1}{2} m v^2 = m g h.$$

Whence $v^2 = 2 g h$ as above.

If a is the area of the orifice, the quantity of water flowing out in time t will be $a\ v\ t$ if Torricelli's law is true. The quantity is always smaller then this because the effective area, called the *vena contracta*, is smaller than the actual orifice.

87. RANGE OF JETS.

Let the distances A B, B C, C D, Fig. 32, be equal to each other and equal to h. Let a stream issue from A with a velocity v. Then the range a of the stream will be determined by the equation

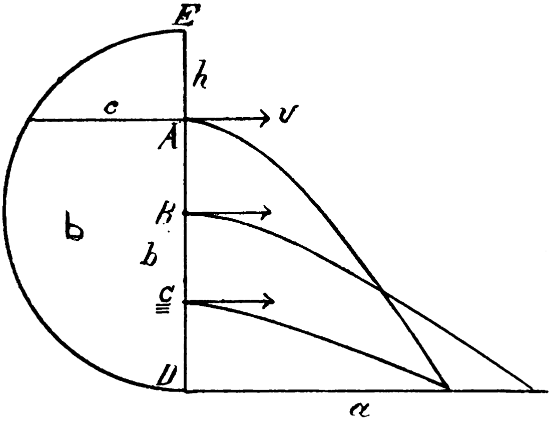


FIGURE 32.

$$a = v\ t.$$

Also $b = \tfrac{1}{2} g t^2.$

Whence $b = \dfrac{a^2}{4 h},$

or $4 b h = a^2$ and $a = 2\sqrt{b h}.$

It follows that if b and h exchange values the range will remain the same. Since the sum of b and h is a constant their product is a maximum when they are equal to each other. Hence the greatest range will belong to the orifice at half the height.

88. THE COMMON SIPHON.

Let y be the height of a cross-section of the siphon at its highest point above the surface of the liquid in the vessel; and let x be the height of the same section above the open end of the siphon. Let H be the atmospheric pressure. Then the pressure on the cross-section outward is $H - y$; and the pressure inward is $H - x$. The difference is the effective pressure producing motion.

$$(H - y) - (H - x) = x - y = h,$$

the effective "head of water."

89. MARIOTTE'S FLASK.

This is a device to secure uniform velocity of efflux. Illustrated in lectures.

Nov. 16th

SOUND.

NATURE AND MOTION OF SOUND.

90. Sound and Hearing.

In all sense perceptions it is necessary to distinguish between the sensations themselves and the external cause of them. Further, the objective causes of our sensations bear no *resemblance* to the sensations themselves. Hence *sound* should be distinguished from *hearing* in the study of sound as a branch of Physics. All questions concerning sound come ultimately for decision to the organ of hearing; but in referring our sensations to their external cause, we are only interpreting signs presented to us and drawing conclusions respecting outward phenomena. When we have discovered the physical facts constituting the foundation of sound, our investigations are largely transferred to the domain of Mechanics.

91. The Source of Sound a Vibrating Body.

Very cursory examination serves to show that the source from which sound proceeds is a vibrating body. "Sound and movement are so correlated that one is strong when the other is strong, one diminishes when the other diminishes, and the one stops when the other stops."*

Any regular succession of taps produces a musical sound. This may be illustrated by the rotation of a heavy toothed wheel or by a circular saw.

The vibrating body may be solid, liquid, or gaseous. Only the first and the last are used in musical instruments.

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* Blaserna on Sound, page 7.

92. THE MEDIUM OF PROPAGATION.

The vibrations of a sonorous body cannot affect the organs of hearing without a medium of communication between them. If the vibrating body be isolated so that no elastic medium can propagate its motions to the ear, no sound will be perceived. This is imperfectly accomplished by a bell under an exhausted receiver. The transmission of sound requires a medium both elastic and ponderable,—elastic, because elasticity is the agency by which the motion is transmitted from particle to particle, and ponderable because sound is not transmitted through space exhausted of ordinary gross matter.

93. DEFINITION OF SOUND.

Sound consists of those vibrations, propagated through an elastic and ponderable medium, which affect the organs of hearing. "Acoustics has for its object the study of those phenomena which may be perceived by the ear."

94. PROPAGATION OF SOUND. (A. & B., 354.)

The vibratory motions of a sounding body are communicated to the air as the usual medium of transmission. These vibrations are said to be *longitudinal*, that is, in the direction of sound transmission. They are thus distinguished from *transverse* vibrations in which the motions of the particles constituting the wave are at right angles to the direction in which the wave is traveling. The manner in which the motion is transmitted from particle to particle may be illustrated by means of a row of elastic balls lying in contact and resting on two curved rails, the ends elevated. If one ball rolls down the groove and strikes against the first one in line, the motion is communicated through the whole series, and the last one moves up the incline. The *elasticity* of the balls explains the transfer of the motion through the series; but the *energy* of the motion is independent of the elasticity of the conducting medium. It must all be supplied at the origin of the motion.

In order to understand the motion by which sound is propagated let A B, Fig. 33, represent an elastic cylinder,

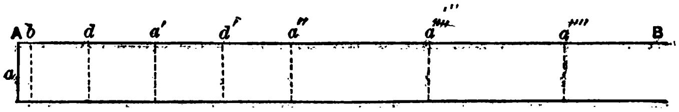

FIGURE 33.

and let the layer a suffer a small displacement to the right. The effect of this displacement is that a approaches b, producing a condensation. Hence b is moved forward and the motion is thus communicated from layer to layer through the cylinder. If now a executes regular vibrations, its motions will be communicated to all the other layers, and in time each layer of particles will be executing vibrations similar to those of a. If the period of the vibration of a is t, and the speed of transmission is v, then in one complete vibration the disturbance will travel a distance $s = v t$, or to a' in the figure. During two complete vibrations it will travel a distance $2s$ or to a''; in three periods to a''', and so on. Hence the layer a' begins its first excursion as a begins its second; a'' begins its first as a' begins its second, and a its third, and so on. The layer at d, midway between a and a' begins its first vibration when a completes its first half vibration, and it therefore moves *forward* while a moves *backward.* This related movement of particles in the cylinder constitutes a *wave.* While a moves forward the particles near it compose a *compression;* while it moves backward they constitute a *rarefaction.* The distance $a\,a'$, $a'\,a''$, etc., is called a *wave-length.*

95. DISTINCTION BETWEEN MOTION OF PARTICLES AND OF THE WAVE. (Desch., 872.)

The motion of the individual particles of the medium conveying sound is entirely distinct from the motion of the sound-wave itself. This is characteristic of all undulatory motion propagated through a medium of motion. A sound-wave is composed of a condensation and a rarefaction. In the former the particles have a forward motion in the direction in which the sound is traveling; in the latter they have a backward motion, while at the same time both condensation and rarefaction are traveling steadily forward with a velocity independent of that of the air particles. This action may be illustrated by an elastic reed vibrating at the mouth of a long tube through which run the pulses produced by the reed. In a sound-wave the motion of the wave and the motion of the particles composing the wave are not identical; a wave in air is in no sense a current; the motion of the condensation and of the particles composing it are in the same direction, while the motion of the rarefaction and the corresponding particles are in opposite directions.

The sound-wave has a continuous, progressive motion, while the air particles only swing back and forth through a narrow range.

VELOCITY OF SOUND.

Since the motion of the parts of the medium through which sound is transmitted and the motion of a sound-wave itself are distinct from each other, the two kinds of motion admit of independent treatment and illustration. Consider first the velocity of sound in air and other media.

96. Experimental Determination of the Velocity of Sound in Air.

The usual determination of the velocity of sound is founded upon the measurement of the interval which elapses between the observation of some phenomenon by sight and by hearing. The observations have generally been those of the flash and the report of a distant cannon.

The following are some of the most reliable results, all at 0° C.:—

1.	Academie des Sciences, 1738	332.0	Meters.
2.	Benzenberg, 1811 (mean)	333.0	"
3.	Goldingham, 1821	331.1	"
4.	Bureau des Longitudes, 1822	330.6	"
5.	Moll and Van Beck, 1822	332.2	"
6.	Stampfer and Myrbach	332.4	"
7.	Bravais and Martin, 1844	332.4	"
8.	Wertheim	331.6	"
best 9.	Stone, 1871	332.4	"
	Mean	332	Meters.

In Mr. Stone's observations a cannon was fired and two observers, three miles apart, gave a signal by electricity upon hearing the report.

97. VELOCITY INDEPENDENT OF PITCH AND LOUDNESS.

It is a fact of direct observation that the velocity of sound is independent of pitch and loudness, at least between wide limits. If this were not so then music, played by several instruments at a distance, would reach the listener out of time and hence confused and discordant. Theory indicates that when the aereal disturbances are very violent, so that the changes in density taking place are comparable with the mean density of the air, the velocity is then not independent of the loudness.

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98. THEORETICAL DETERMINATION OF VELOCITY. (Phil. Trans., 1870, 277; Max. Heat, 223.)

Let A_1, A_2, Fig. 34, be a tube of unit cross-section

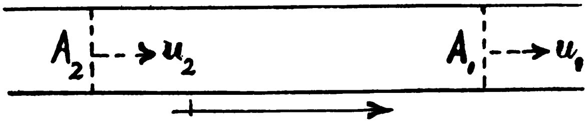

FIGURE 34.

and indefinite length. Let A_1 and A_2 be two imaginary planes traveling with the velocity of sound V. Also let

u_1, u_2 be the velocities of the air particles at A_1, A_2, p_1, p_2 the corresponding pressures, and d_1, d_2 densities at the same places. Hence $V - u_1$ and $V - u_2$ are the velocities of the two planes with respect to the medium; and $(V - u_1)\, d_1$ and $(V - u_2)\, d_2$ are the masses of air traversed by A_1, A_2 respectively in one second. These masses are equal. Hence

$$(V - u_1)\, d_1 = (V - u_2)\, d_2 = V\, d = m.$$

The change of momentum of the mass m in passing from one plane to the other is $m\,(u_2 - u_1)$: this equals the difference of pressure, or

$$p_2 - p_1 = m\,(u_2 - u_1).$$

Substituting for u_1 and u_2 their values obtained from the first equation, and

$$p_2 - p_1 = m^2\left(\frac{1}{d_1} - \frac{1}{d_2}\right).$$

Let volumes containing unit mass of air at densities d_1, d_2 and d be represented by s_1, s_2 and s. Then since these volumes are the reciprocals of the corresponding densities, we have

$$p_2 - p_1 = m^2\,(s_1 - s_2) = V^2 d^2\,(s_1 - s_2).$$

Whence $V^2 = \dfrac{1}{d^2} \cdot \dfrac{p_2 - p_1}{s_1 - s_2} = s^2\, \dfrac{p_2 - p_1}{s_1 - s_2}$.

Let e represent the elasticity of the air, defined as the quotient of the pressure applied by the compression produced. Then

$$e = s\,\frac{p_2 - p_1}{s_1 - s_2}.$$

Therefore $V^2 = e\, s = \dfrac{e}{d}$, and $V = \sqrt{\dfrac{e}{d}}$.

99. TO DEMONSTRATE THAT ELASTICITY EQUALS PRESSURE.

Let P and D represent corresponding pressure and density of the air. Let the pressure be increased by a small quantity p, and let the volume of unit mass of air be diminished in consequence by a quantity s. Then by Boyle's law

$$\mathrm{P} : \mathrm{P} + \mathrm{p} :: \mathrm{S} - \mathrm{s} : \mathrm{S}.$$

Whence $$\mathrm{p} : \mathrm{P} + \mathrm{p} :: \mathrm{s} : \mathrm{S},$$

or $$\frac{\mathrm{p}}{\mathrm{P} + \mathrm{p}} = \frac{\mathrm{s}}{\mathrm{S}} = \text{compression}.$$

Hence elasticity $= \mathrm{p} \div \dfrac{\mathrm{p}}{\mathrm{P} + \mathrm{p}} = \mathrm{P} + \mathrm{p}.$

When the disturbance is small p is negligible in comparison with P, and therefore

$$\mathrm{V} = \sqrt{\frac{\mathrm{P}}{\mathrm{D}}}.$$

In great explosive disturbances p is not negligible. The formula shows that the speed of sound is then greater than for small disturbances.

100. NEWTON'S FORM OF THE EQUATION. (Desch., 883.)

From section 81, $\mathrm{P} = \mathrm{H\,d\,g}$, where H is the height of the homogeneous atmosphere.

Hence $$\frac{\mathrm{P}}{\mathrm{D}} = \mathrm{g\,H}.$$

Therefore $$\mathrm{V} = \sqrt{\mathrm{g\,H}}.$$

If a heavy body fall through a height H, the velocity attained is determined by the equation

$$\mathrm{v} = \sqrt{2\,\mathrm{g\,H}}.$$

The speed of sound is therefore equal to the velocity attained by a body falling *in vacuo* through half the height of the homogeneous atmosphere. This is the form in which Newton announced the result of his investigation.

101. THE VELOCITY OF SOUND AS AFFECTED BY CHANGES OF TEMPERATURE.

The effects upon the velocity of sound in air, due to changes in temperature arising from two distinct causes, must be distinguished from each other. One change is that of the average temperature of the mass of air through which the sound passes; the other is that due to compression and rarefaction in the two complementary portions of a sound-wave. The former may be considered as affecting simply the pressure P; the latter as augmenting the elasticity independently of pressure. First, the effect upon pressure of a change in temperature. Let t represent the temperature and a the coefficient of expansion. Then a pressure P at 0° C. will become $P(1 + a\,t)$ at t° C., density remaining constant. Hence the expression for velocity becomes

$$V = \sqrt{\frac{P_0(1 + a\,t)}{D}}.$$

$V = V_0 + .6t$

The value of a is 0.003663. Therefore the velocity is increased by the factor $\sqrt{1.003663} = 1.00181$ for every degree C. Hence the increase per degree equals $332 \times 0.00181 = 0.6$ meter, or nearly two feet.

Second, the effect due to compression and rarefaction. Elasticity equals pressure only under the condition 'applying to Boyle's law—that the temperature remain constant. But when a sound wave passes, the compressions and rarefactions follow each other so rapidly that changes in temperature due to them have no time to become equalized. Hence the elasticity to be considered is that corresponding to expansion and compression without heat entering or escaping—adiabatic expansion. The ratio of this elasticity to that at constant temperature is 1.41. Therefore the complete formula for velocity in air becomes

$$V = \sqrt{1.41\,\frac{P(1 + a\,t)}{D}}.$$

102. COMPUTATION OF THE VELOCITY OF SOUND.

The following is the computation in C. G. S. units for dry air at 0° C: The pressure P is that due to a column of mercury 76 cm. high and of one sq. cm. cross-section, or $76 \times 13.59 \times 981 = 1013373$ dynes per sq. cm.

D equals 0.001293 gms. at 0°. Hence,

$$V = \sqrt{\frac{1.41 \times 1013373}{0.001293}} = 33240 \text{ cm.} = 332.4 \text{ meters.}$$

103. VELOCITY OF SOUND IN WATER. (Desch., 883.)

By the experiments of Colladon and Sturm, made in Lake Geneva, the velocity of sound in water was found to be 1435 meters per second. The general formula gives

= coeff. of volume elasticity

$$V = \sqrt{\frac{E}{D}} = \sqrt{\frac{2.1 \times 10^{10}}{1}} = 144914 \text{ cm.}$$

or 1449.14 meters.

104. VELOCITY OF SOUND IN SOLIDS. (Desch., 884.)

Applying the general formula to copper,

= Young's modulus [illegible]

$$V = \sqrt{\frac{120 \times 10^{10}}{8.8}} = 369300 \text{ cm. or } 3693 \text{ meters per sec.}$$

This is about 11.1 times as great as in air. Wertheim found it to be 11.167 times as great by experiment.

105. RELATIONS BETWEEN VELOCITY, WAVE-LENGTH, VIBRATION-FREQUENCY AND PERIOD.

If *n* equal the number of complete oscillations of an air particle per second, λ the wave-length or the distance the sound travels while the sonorous body makes one complete vibration, and T the period of a complete vibration; then, since λ is the length of one wave, and *n* such waves proceed from the source in one second,

$$\lambda n = V, \text{ or } \lambda = \frac{V}{n}.$$

Further, since the vibration-frequency *n* and the period T are reciprocals of each other,

$$\lambda = V\,T.$$

MOTION OF THE MEDIUM.

106. Simple Harmonic Motion Applied to Sound.

When a tuning fork is vibrated and left to itself the loudness of the sound diminishes but the pitch remains constant. The constancy of pitch indicates constancy of vibration-frequency; for the pitch of a sound depends only on the period of the vibrations which constitute its physical cause. Unless an elastic body be very widely distorted, its periodic time, and therefore the pitch of the sound produced, are independent of the amplitude of vibration. The proportionality of the elastic forces, called into action by displacement, to the displacement itself is a fundamental law of the vibratory motions which give rise to musical sounds. Hence the vibrations of the parts of musical instruments, as well as the swing of the air particles to which they give rise, may all be studied under the conditions applying to Simple Harmonic Motion.

107. Wave Motion as a Curve of Sines. (A. & B., 356; Everett's Vibratory Motion and Sound, 46).

If the S. H. M. of an air particle be assumed to take place at right angles to the uniform motion of the wave, a sinusoidal curve may then be constructed to represent either the displacement of successive particles or their velocities at a given time. If *a* is the amplitude of motion and *x* the displacement of the particle, then the two equations of motion are

$$x = a \cos\left(\frac{2\pi t}{T} - \varepsilon\right);$$

$$y = V\,t.$$

If t is reckoned from the position of equilibrium,

$$\varepsilon = \frac{\pi}{2},$$

and
$$x = a \sin \frac{2 \pi t}{T}.$$

Substituting for t its value $\frac{y}{V}$ from the second equation,

$$x = a \sin \frac{2 \pi y}{T V} = a \sin \frac{2 \pi y}{\lambda} \text{ (section 105).}$$

Hence x is a periodic function of y. The same value of x recurs with every increase in y equal to λ; λ is called the wave-length. If various values are given to y the corresponding values of x will represent the displacements of the particle the distance of which from the origin is y. For $y = 0, x = 0$; for $y = \frac{1}{4} \lambda, x = a$; for $y = \frac{1}{2} \lambda, x = 0$; for $y = \frac{3}{4} \lambda, x = - a$; and for $y = \lambda, x = 0$. Laying off the values of y along a straight line and erecting perpendiculars equal to the corresponding values of x the curve of sines is obtained. Similarly the formula for the curve of velocities is

$$x = \frac{2 \pi a}{T} \cos \frac{2 \pi y}{\lambda}.$$

From the two equations it is evident that maximum velocity is simultaneous with minimum displacement and *vice versa*.

Of the two curves in Fig. 35 the upper represents displacements and the lower velocities.

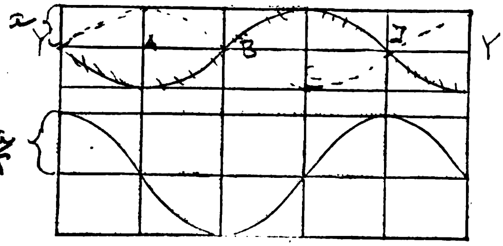

FIGURE 35.

heavy line meaning?

y	x
0	$\frac{2\pi a}{T}$
$\frac{1}{4}\lambda$	0
$\frac{1}{2}\lambda$	$-\frac{2\pi a}{T}$
$\frac{3}{4}\lambda$	

108. COMPOSITION OF S. H. M'S IN THE SAME PLANE. (A. & B., 359.)

The curve of sines may be used to explain the composition of two or more wave motions in the same plane. If two systems of waves coexist in the same medium the displacement at any point will be the sum of the displacements due to the two systems taken separately. Hence the actual displacements may be found by taking the algebraic sum of the ordinates of the two displacement curves. If the two systems have the same period, then the resulting curve will be a simple sine curve of the same period; if the periods are not the same, then the composition curve will not be a curve of sines.

In Fig. 36 the dotted and the light lines represent the displacements due to two wave systems of the same period and amptitude. The heavy line represents the resulting displacement. In I the two systems have the same phase, and the resulting amptitude is double that of either component. In II the phases differ by one-quarter, and in III by one-half a period. In the last case the two motions

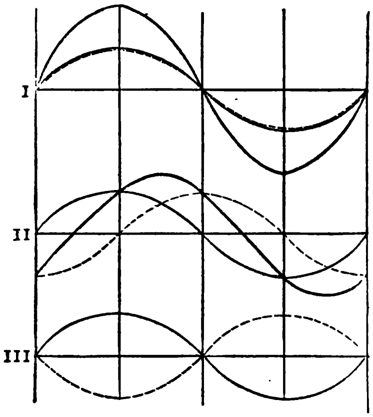

FIGURE 36.

completely annul each other. In Fig. 37 the periods of the

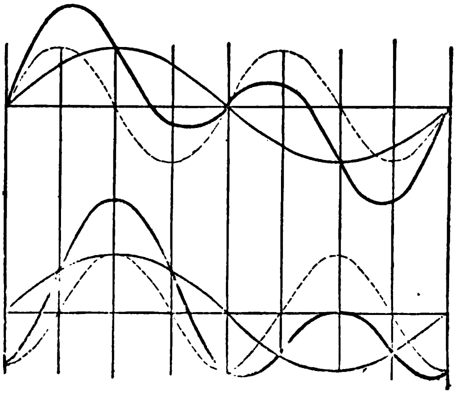

FIGURE 37.

two wave systems are as 1 : 2. The resulting curve is not a sinusoid.

109. INTERFERENCE AND BEATS. (Desch., 890–893.)

When the two component sound-waves have nearly the same period the case deserves special attention. The reciprocal action results in *interference and beats.* For a few vibrations the periods may be regarded as the same, and the resulting vibration will be simple harmonic. But the more rapid vibration will gain on the other, thus changing the difference of phase on which the resultant depends. Suppose the two systems, of nearly the same amptitude, have vibration-frequencies m and n, where $m - n$ is very small. Let the phases agree initially. Then after an interval of $\frac{1}{2}/(m - n)$ seconds the phases will be exactly opposite, one system will have gained half a wave-length on the other, and almost total extinction of motion and sound will ensue.

After a further equal interval of time the system of shorter period m will have gained one vibration on the other, complete agreement will once more be established, and an increase of sound will be observed. This phenomenon, due to interference, is known as *beats*. The number of such re-enforcements of sound per second is equal to the difference in the vibration-frequencies of the two notes, or $m - n$.

110. TO COMBINE TWO S. H. M'S OF THE SAME PERIOD AT RIGHT ANGLES TO EACH OTHER.

The equations of the two are

$$x = a \cos \theta,$$

$$y = b \cos (\theta - \epsilon) = b (\cos \theta \cos \epsilon + \sin \theta \sin \epsilon).$$

From the first, $\cos \theta = \frac{x}{a}$; then $\sin \theta = \frac{1}{a} \sqrt{a^2 - x^2}$.

$$\text{Hence, } y = \frac{b}{a} (x \cos \epsilon + \sqrt{a^2 - x^2} \sin \epsilon);$$

$$\text{or,} \quad (y - \frac{b}{a} x \cos \epsilon)^2 = \frac{b^2}{a^2} (a^2 - x^2) \sin^2 \epsilon.$$

This is an equation of the second degree; and since the curve returns into itself, or is a closed curve, it must be an ellipse. Consider four cases.

(1) When $\epsilon = 0$. Then $\cos \epsilon = 1$; $\sin \epsilon = 0$.

Therefore $y - \frac{b}{a} x = 0$.

This is the equation of a straight line.

(2) When $\epsilon = \frac{\pi}{2}$. Then $\cos \epsilon = 0$; $\sin \epsilon = 1$.

Therefore $y^2 = \frac{b^2}{a^2} (a^2 - x^2)$.

or $a^2 y^2 + b^2 x^2 = a^2 b^2$.

This is the equation of an ellipse whose semi-axes are a and b.

(3) When $\epsilon = \pi$. Then $\cos \epsilon = -1$; $\sin \epsilon = 0$.

Therefore $y + \frac{b}{a} x = 0$.

This is again the equation of a straight line.

(4) When $\varepsilon = \frac{3\pi}{2}$. Then $\cos\varepsilon = 0$; $\sin\varepsilon = -1$.

Therefore $y^2 = \frac{b^2}{a^2}(a^2 - x^2)$.

or $a^2 y^2 + b^2 x^2 = a^2 b^2$,

the equation of an ellipse, as before, referred to its center and axes.

If a rectangle be drawn with sides equal to twice the amplitudes *a* and *b*, then all the ellipses that can be obtained by varying ε will lie within the rectangle and touch its sides.

111. TO COMBINE TWO S. H. M'S AT RIGHT ANGLES WITH PERIODS AS ONE TO TWO.

Since $\theta = \frac{2\pi t}{T}$ and $\theta' = \frac{2\pi t}{T'}$, if T′ is one-half T, $\theta' = 2\theta$. Therefore the two equations are

$$x = a \overset{\text{sin}}{\cos}\theta,$$

$$y = b \overset{\text{sin}}{\cos}(2\theta - \varepsilon).$$

These will give an equation of the fourth degree; but if $\varepsilon = 0$, the equation reduces to one of the second degree.

For $\frac{x}{a} = \cos\theta$; $\frac{y}{b} = \cos 2\theta = 2\cos^2\theta - 1$.

Hence $\frac{y}{b} = 2\frac{x^2}{a^2} - 1$,

and $x^2 = \frac{a^2}{2b}(y + b)$.

This is the equation of a parabola "whose vertex is at a distance *b* from the central point of the vibrations, and focus at the distance $\frac{a^2}{8b}$ from the vertex."

112. GRAPHICAL METHOD OF COMBINING S. H. M'S AT RIGHT ANGLES.

Draw two concentric circles with radii equal to the two amplitudes of motion. Divide the circles into equal parts, some multiple of four, which shall be proportional to the two periods of vibration. Through these divisions of the circles draw parallel lines, the sets for the two circles being at right angles. The distances between these parallel lines equal the motions due to the two systems for equal successive time intervals; and since the same material point is subject to the two motions, it must be on successive *intersections* of these parallel lines at the ends of successive equal intervals. Any successive pair of intersections traversed will lie at the diagonally opposite corners of the small rectangles made by the parallel lines. A curve traced through the points will represent the resulting motion. Thus in Fig. 38 the periods are as 2 to 3 and the circles are divided into 8 and 12 parts respectively. The curve repre-

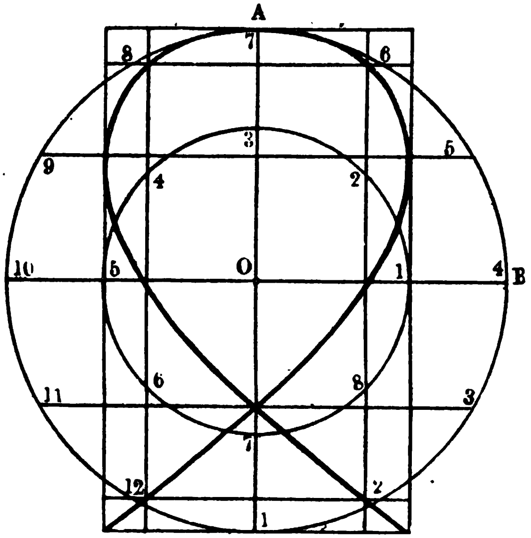

FIGURE 38.

sents the case of no difference in phase. The results obtained analytically are confirmed by Blackburn's pendulum and Lissajou's forks.

PHENOMENA OF REFLECTION.

113. THE PRINCIPLE OF HUYGHENS. (D., 106; A. & B., 356.)

Let *a*, Fig. 39, be a center or source of spherical sound waves. The principle of Huyghens is that every point of the wave-surface *m n* becomes a new center of disturbance from which waves of sound are propagated in the same manner as from the original center. This follows from the consideration that every particle on the wave-surface has the same vibratory motion, except in respect to amplitude, as the first particle disturbed, and therefore stands in the same relation to adjacent particles and communicates motion to them in the same manner. It therefore becomes itself a center of disturbance. Let the points of the surface *m n* be such centers from which waves proceed for a short distance *c d*. The number of such waves being indefinitely large, they will ultimately coalesce to form the new wave-surface *m′ n′* which is tangent to all the elementary wave-surfaces.

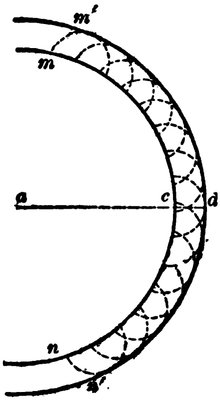

FIGURE 39.

114. REFLECTION OF A PLANE WAVE AT A PLANE SURFACE. (D., 112.)

Let A B, Fig. 40, be a portion of the plane advancing wave, and let C D be the reflecting surface. If A B had met with no obstruction it would have taken the position A′ B′ when B had reached B′. But A becomes a new center of disturbance which travels backward in the first medium a distance A A′. Hence with A as a center and with radius A A′ equal to B B′ draw a circle. So *b* would have reached *b′* in the same time; but it proceeds to E and is then reflected. Hence draw about E a circle with radius E *b′*. Draw any number of circles in the same manner. Then from B′ draw a tangent to the first circle; it will evidently touch all the other circles and will be the reflected wave-front. Draw A A″ to the point of tangency. Then the triangles A B′ A″ and A B B′ are equal to each other and the angles B A B′ and A B′ A″ are equal. But these are the angles of incidence and reflection respectively.

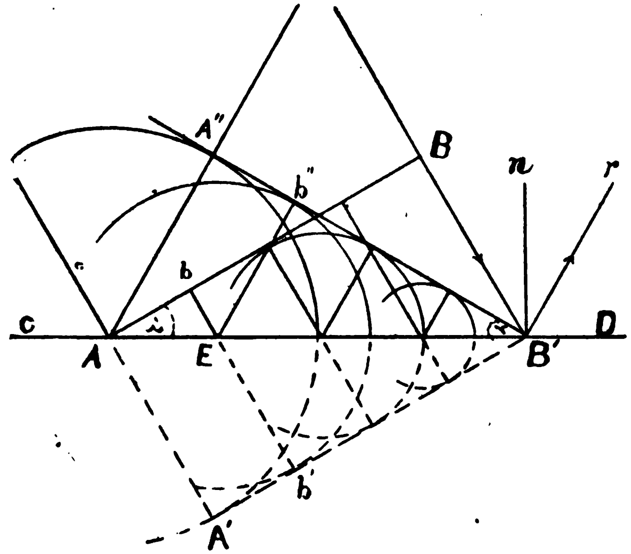

FIGURE 40.

115. RELATION OF THE CENTERS OF THE DIRECT AND REFLECTED SYSTEMS OF WAVES. (Desch., 886.)

Let O, Fig. 41, be the center of the incident waves which are reflected from the surface A B. If these waves had met with no obstruction they would have taken positions at equal successive time-intervals indicated by the dotted lines; but they are reflected so as to have the positions of the full lines symmetrically situated on the other side of the reflecting surface. Let O I be a sound ray or the direction of motion of any point of the incident wave; draw I M so that I M and O I shall make equal angles with the normal. Then is I M the path of the reflected ray, that is, it is a normal to the reflected waves. Project I M backward till it intersects the normal to the reflecting surface through O. Then is O′ the center of the reflected waves. The triangles O I C and O′ I C are equal. Hence O C and O′ C are equal, and the centers of the incident and reflected waves are on a perpendicular to the reflecting surface and equidistant from it. The sound center and sound image are symmetrically situated with respect to the reflecting surface.

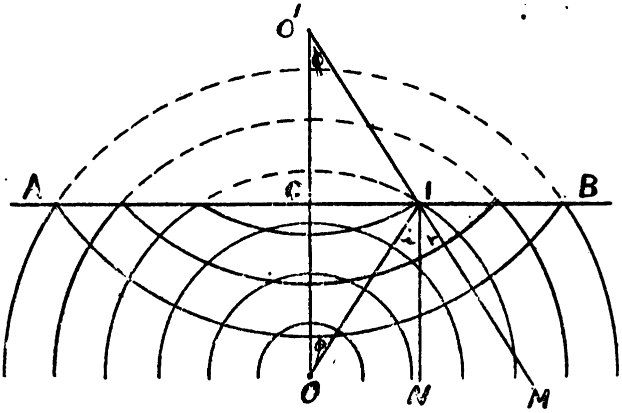

FIGURE 41.

116. ECHO. (Desch., 887; D., 412.)

The most important illustration of the reflection of sound-waves occurs in the case of echoes. It appears that the sensation of sound persists for about one-tenth of a second, during which time sound travels about 34 meters.

If then the distance of the reflecting surface exceeds 17 meters the observer may hear the reflected sound separately. This repetition of a sound by reflection is called an echo. Parallel reflecting surfaces at a suitable distance produce multiple echoes. Reflection of sound takes place from buildings, rocks, woods and even damp sails of vessels.

PHYSICAL THEORY OF MUSIC.

117. Musical Intervals.

When two notes are sounded together or in quick succession the ear recognizes a special relationship existing between them, involving a perception of their relative pitch.

This relationship is expressed as a ratio between their vibration-frequencies and is entirely independent of the absolute pitch of the notes.

Many of these ratios have definite names. Thus the ratio of one to one is called *unison;* of $\frac{2}{1}$, an *octave;* of $\frac{3}{1}$, a *twelfth;* $\frac{3}{2}$, a *fifth;* of $\frac{4}{3}$, a *fourth;* $\frac{5}{4}$, a *major third;* of $\frac{6}{5}$, a *minor third.* Numerically an interval is always equal to or greater than unity.

118. The Diatonic Scale or Gamut.

When three notes, whose vibration-frequencies are as 4 : 5 : 6, are sounded together a pleasing effect is produced upon the ear. Such a combination of notes is called a *major triad;* and together with the octave of the lowest of the three they constitute a *major chord.* The perfect diatonic scale is built on three sets of such triads.

If the three notes have vibration-frequencies as 10 : 12 : 15, they constitute a *minor triad;* and with the octave of the lowest, a *minor chord.* The resulting curves for the two triads are shown in Fig. 42.

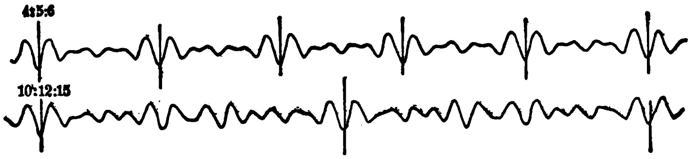

FIGURE 42.

The eight notes of the scale are represented by the letters C, D, E, F, G, A, B, C.

The three major triads are

$$\left.\begin{array}{l} C:E:G \\ G:B:d \\ F:A:c \end{array}\right\} :: 4:5:6.$$

When the several notes of the scale are thus related they are found to give the most agreeable chords.

If then C is due to m (about 64) vibrations per second, the vibration-frequencies of the other notes of the scale may be found by simple proportion from the above relations.

$$\text{Thus}\quad C:E::4:5. \quad \text{Hence } E = \frac{5}{4}C = \frac{5}{4}m.$$

$$C:G::4:6. \quad \text{"} \quad G = \frac{3}{2}C = \frac{3}{2}m.$$

Pursuing this method the following numbers are found to represent the relative vibration-frequencies of the several notes:—

C,	D,	E,	F,	G,	A,	B,	C.
1	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{15}{8}$	2.

Reducing to a common denominator the numerators are

24, 27, 30, 32, 36, 40, 45, 48.

The successive intervals are accordingly

C to D	D to E	E to F	F to G	G to A	A to B	B to C
$\frac{9}{8}$;	$\frac{10}{9}$;	$\frac{16}{15}$;	$\frac{9}{8}$;	$\frac{10}{9}$;	$\frac{9}{8}$;	$\frac{16}{15}$.

There are thus only three different intervals from note to note through the perfect diatonic scale. The intervals $\frac{9}{8}$ and $\frac{10}{9}$ are called whole tones, and $\frac{16}{15}$ a half tone or a *limma.*

The difference between the first two is $\frac{81}{80}$ called a *comma.* The intervals from C to each of the other notes in order are called a second, a major third, a fourth, a fifth, a major sixth, a seventh, and an octave.

119. MINOR CHORDS AND TRANSITION.

The interpolated notes additional to the eight of the diatonic scale are rendered necessary in order to provide for minor chords and to be able to pass from a scale in one key to that of another, a process which is called *transition.* The middle note of a minor triad is lower than that of a major by an interval of $\frac{25}{24}$. Three interpolated notes become necessary in the key of C, viz., three notes below E, B, A, by the above interval.

But if we suppose the gamut to begin with G, then the seven other notes must follow with the same succession of intervals as in the key of C, that is, $\frac{9}{8}$, $\frac{10}{9}$, $\frac{16}{15}$, etc. Or in other words for the key of G, the three sets of major triads are

$$\left.\begin{array}{l} G : B : d \\ D : F : A \\ C : E : G \end{array}\right\} :: 4 : 5 : 6.$$

Comparing these with the triads for the key of C it will be seen that two of them are identical, while the third contains two notes, F and A, differing from the scale in the key of C. A numerical comparison of the two scales shows the exact difference.

Key of C.

c′, d′, e′, f′, g′, a′, b′, c″, d″, e″, f″, g″,
256, 288, 320, 341.3, 384, 426.7, 480, 512, 576, 640, 682.6, 768

Key of G.

256, 288, 320, 360, 384, 432, 480, 512, 576, 640, 720, 768

The interval between the a″s of the two scales is $\frac{81}{80}$, while the interval between the f's is much larger. One other new note besides these two is necessary to provide for minor triads. But other keys are employed also, some introducing a still larger number of extra notes; so that, with all the naturals as key-notes, the scale would comprise about 100 distinct tones.

Dec 7th

120. Tempered Scales. (D., 381.)

The number of notes required to provide for scales in all keys is far in excess of possible provision in an instrument with fixed keys. Hence some system of accommodation must be adopted by which the number of notes shall be much reduced. Such modification of the notes is called *tempering.* Every system of tempering changes slightly the pitch of each note, so as to bring together into one all the tones falling between the eight of the diatonic scale. The intervals from E to F and from B to A being already semi-tones, no notes are interpolated there. Hence the extra notes occur in groups of threes and twos, and there are thirteen notes in the scale.

The system applied to the organ and piano is known as that of "equal temperament." It ignores all differences between major and minor tones and makes the successive intervals all equal; and since the octave is divided into twelve equal intervals, each one equals $\sqrt[12]{2}$ or 1.05946. The result differs widely from pure intonation. Such a scale of equal semi-tones is called the *chromatic* scale.

STRINGS, PLATES AND PIPES.

121. TRANSVERSE VIBRATIONS OF STRINGS. (D., 909.)

When a disturbance is produced at any point of a stretched string it runs in both directions to the fixed ends of the string from which it is reflected, and passing back on the opposite side is again reflected and finally arrives at the starting point. The string has then returned to its initial position, it has executed one complete vibration, and each half of the pulse has traversed the length of the string twice. Hence if l is the length of the string and λ the wave-length of its fundamental tone, $\lambda = 2\,l$.

122. LAWS OF THE TRANSVERSE VIBRATION OF STRINGS. (D., 388; A. & B., 375; Desch., 909.)

The expression for the speed of the pulse along the string is $v = \sqrt{\frac{k}{d}}$, in which k is the tension in dynes per sq. cm. of cross-sectional area, and d is the density of the string. If T is the stretching weight in grammes, $k = \frac{\mathrm{T}\,g}{\pi\,r^2}$, r being the radius of the string. Hence

$$\mathrm{v} = \sqrt{\frac{\mathrm{T}\,g}{\pi\,r^2\,d}}.$$

If we put t for T g, and m for $\pi\,r^2\,d$, or mass per unit length, then $v = \sqrt{\frac{t}{m}}$.

Hence $$n = \frac{v}{\lambda} = \frac{1}{2\,l}\cdot\sqrt{\frac{t}{\mathrm{m}}}.$$

Whence the following laws:—

1. The vibration-frequency varies inversely as the length of the string, when the length alone is changed.

2. The vibration-frequency is directly proportional to the square root of the tension.

3. Strings of the same length, stretched with the same forces, vibrate with frequencies inversely as the square root of their masses per unit length.

4. Strings of the same length and density will have the same period of vibration if they are stretched with forces proportional to their cross-sectional areas.

123. SEGMENTAL VIBRATION OF STRINGS. (Tyndall on Sound, 93.)

A string is capable of vibrating not only as a whole, but also in equal segments; and the number of such segments when it is made to vibrate in a single mode, depends upon the periodic time of the disturbances communicated to it. If a stretched string or wire is lightly touched at its middle point and one half is thrown into vibration, the pulses will run across the middle and will excite vibrations of like period in the other half. If one-third be vibrated the disturbances will be transmitted across the point touched, and will succeed one another at such intervals that the remaining two-thirds of the string will vibrate in two parts, so as to synchronize with the third first vibrated; and so on with higher divisions.

The points separating the equal divisions of the string are places of minimum motion called *nodes;* the middle points of the segments are places of maximum motion called *antinodes.* The small amount of motion transmitted across the nodes accumulates so as to produce wide amplitude at the antinodes. Any two adjacent segments are always moving in opposite directions.

124. OVERTONES.

Whenever a stretched string is set vibrating it vibrates not only as a whole, giving its fundamental tone, but at the same time it divides into one or more sets of equal segments; so that usually there are several series of vibrations superposed upon the fundamental. The tones of higher pitch associated with the fundamental are known by the general name of *overtones* or *upper partials.* In the case of a string the division will be into 2, 3, 4, 5, etc., equal segments, with vibration-frequencies 2, 3, 4, 5, etc., times that of the fundamental tone. The intervals between the fundamental and the overtones are therefore, the octave, the octave plus the fifth (or the twelfth), the double octave, two octaves plus the third, two octaves plus the fifth, etc.

125. DISTINCTION BETWEEN PARTIAL TONES AND HARMONICS. (Koenig's Quelques Expériences d'Acoustique, 218; Wiedemann's Annalen, 1881.)

The vibrating parts of musical instruments may execute a great variety of complex motions; but the vibrations, in order to produce musical sounds, must be periodic; that is, they must repeat themselves in certain definite and equal intervals. Such periodic motions are subject to the following law of Fourier:— *Every complex periodic motion may be resolved into a definite number of commensurate S. H. M's.* The periods of the simple harmonic components of the complex periodic motion are all exact multiples of the fundamental. These components will have definite periods, amplitudes, and phases. "Among the sounds into which the sonorous mass, which emanates from a vibrating body, may be resolved, we may distinguish harmonics and partial tones. These last have their origin when the body in question executes simultaneously several modes of vibration which it can adopt separately, as in the case of the string; while the harmonics are due to the resolution into simple pendular motions of the complex periodic motion of the sounding body executing a single mode of vibration." Partial tones, therefore, are due to the actual subdivision of the sonorous body into segments; harmonics on the contrary are the commensurate components of the motion when the body vibrates periodically by one mode in a complex way. Harmonics are exact multiples of the fundamental; partial tones may or may not be exact multiples. Harmonics represent always the series of whole numbers in all their purity, while the vibration-frequencies of partial tones approach only more or less nearly to their theoretical value.

126. TRANSVERSE VIBRATION OF PLATES. (Koenig, 32; Comtes Rendus, Mar. 27, 1864.)

If a long slender plate be made to execute free transverse vibrations, it will exhibit nodes at a little less than one-quarter of its length from each end. As the ends are bent round nearer and nearer together, as in a tuning fork, the nodes approach the center. If now a thin square plate be vibrated, in general two series of periodic motions will be superposed with nodal lines at right angles as shown in Fig. 43. These two systems may coexist in two ways. If the two systems are related as shown in Fig. 43, then the resulting nodal lines will be those of 3, Fig. 44. If, however, the motions of one of the two systems of Fig. 43 be reversed, the resulting system of nodal lines will be those of 4 in Fig. 44. This last represents the simplest division of a square plate clamped at its center and bowed

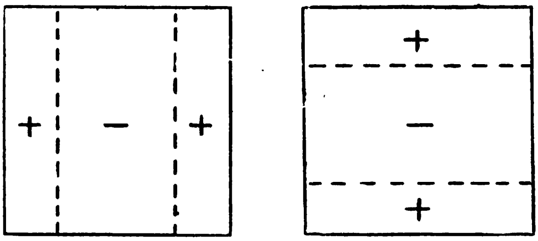

FIGURE 43.

so as to give its fundamental tone. Similarly a round plate clamped at its center divides into four equal segments

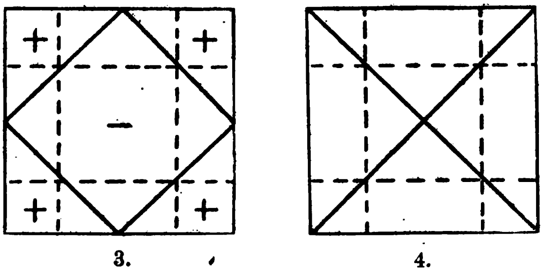

FIGURE 44.

for its fundamental. The next higher tone will be due to the division of the plate into six segments by three diameters, etc. Adjacent segments are always in opposite phases of motion. Figures produced by sprinkling sand on such plates and vibrating them are called Chladni figures.

127. RESONANCE.

Resonance is a particular case of sympathetic vibration, which depends upon the cumulative effect of small disturbances when applied to a body in such a way as to synchronize with its own movements. One wire thus takes up the vibrations of another so conditioned as to vibrate in the same time. So with two pendulums, clocks, bridges, windows, tuning forks, etc. The two bodies may be of different character and still covibrate. Thus an organ pipe may set a tuning fork vibrating if it is mounted on a proper resonant box. Every elastic body has its own rate of vibration, depending upon its elasticity, density, and dimensions. Hence every mass of confined air has a definite rate of vibration. The increase in the volume of sound, depending upon the synchronous vibration of another body, such as a mass of partly enclosed air, is called *resonance*. The resonators of Helmholtz are of great service in the analysis of complex sounds, because each one selects for reenforcement the particular component corresponding with its own predetermined rate, and thus this tone is differentiated from all the rest. The "sound of the sea," heard in a sea-shell, is an illustration of reenforcement by selective resonance.

128. RELATION BETWEEN THE LENGTH OF ORGAN PIPES AND THE WAVE-LENGTH OF THEIR FUNDAMENTAL TONE.

Suppose a reed vibrating at the middle of a closed pipe, Fig. 45, and that its rate corresponds with that of the enclosed air. While the reed moves from *a″* to *a* the condensation runs from it to the end A and is reflected, meeting the reed in the position *a′*.

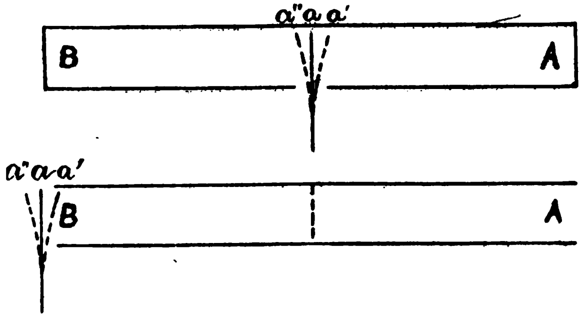

FIGURE 45.

Similarly during the return movement of the reed from *a′* to *a″* the condensation runs to the end B and back to the middle. Thus the reed has executed one vibration, the condensation has traversed the pipe twice, and the two are ready to repeat the process. The motion of the rarefaction may be followed in a similar way. The length of the pipe is therefore half the length of the sound-wave produced in the air by the reed.

If now the pipe be cut in two at the middle and the reed be set vibrating at the open end, the condensation will first run through the pipe and back, followed by the rarefaction, which also traverses the pipe twice, when the initial conditions recur. The disturbance travels four times the length of the pipe during a complete vibration of the reed. The closed organ pipe is therefore one-fourth the wave-length of its fundamental tone in air.

If the reed be supposed to vibrate at one end of an open pipe, then while the reed moves from *a''* to *a'* the condensation runs the entire length of the pipe and is then reflected from the open end as a rarefaction. While the reed moves back from *a'* to *a* rarefactions run in from both ends of the pipe and meet at the middle, producing there a node. These will be followed by two condensations meeting and undergoing reflection at the same place. Thus it will be seen that the disturbance travels twice the length of the pipe during a complete vibration of the exciting body, and that a node is always formed at the middle when a pipe is giving its fundamental tone. The open pipe is thus half as long as the wave in air of its lowest tone. If a stopped pipe and an open pipe give notes of the same pitch the open pipe is twice the length of the closed one. A node is a place of minimum motion and maximum change of density; while an antinode is a place of maximum motion of the air column and minimum change of density.

129. RELATION OF OVERTONES TO FUNDAMENTALS IN OPEN AND IN CLOSED PIPES. (D., 920 and 922.)

The overtones produced by organ pipes are due to a division of the vibrating column of air into segments, the distances between the nodes and antinodes all being equal. Thus in the closed organ pipe the divisions for successive overtones will be as represented in Fig. 46. The vibration-

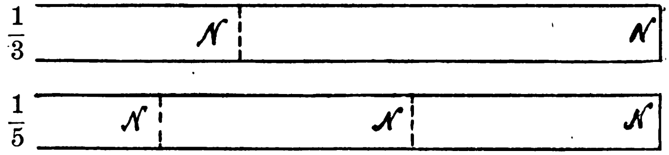

FIGURE 46.

frequencies are therefore represented by the odd numbers 1, 3, 5, 7, etc., the lengths of the vibrating-segments being as 1, $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{7}$, etc, both including the fundamental tone.

For open pipes a node always occurs at the middle for the fundamental. The addition of nodes for successive overtones will result in divisions of the pipe as shown in Figure 47. Hence the relative lengths of the half-segments

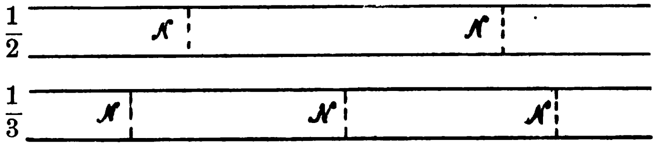

FIGURE 47.

are 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, etc., and the relative vibration-frequencies 1, 2, 3, 4, etc.

FUNCTION OF OVERTONES.

130. BEATS DUE TO OVERTONES. (Desch., 942; A. & B., 385.

Beats are produced not only between two notes nearly in unison, but between notes whose interval is *approximately* an octave, a major third, a fifth and so on. These Helmholtz attributes to the overtones associated with the fundamentals. Thus if two notes have vibration-frequencies of n and $2n + 1$, then the first overtone of the lower will be due to $2n$ vibrations per second, and this will produce one beat per second with the higher note. So also if two notes due to $2n + 1$ and $3n$ vibrations per second respectively, then the second overtone of the first will be due to $6n + 3$, and the first overtone of the second to $6n$ vibrations per second, giving three beats per second, though the interval is otherwise indistinguishable from a fifth. Combinations of such vibrations, obtained mechanically by Kœnig, show periodic variations of amplitude corresponding with the beats.

Again, the interval between the fundamentals may be exact, but the overtones may be partial tones, and so not exact multiples of the fundamentals. Such is the case with tuning forks, and beats are heard between their overtones of the same order.

131. The Quality of Sound.

The three essential characteristics of musical sounds are pitch, intensity, and quality or *timbre.* The first depends upon the *wave-length*, the second upon *amplitude* of vibration, and the third upon the only other remaining point upon which aerial waves can differ, viz., their *form.* By form is meant the law according to which the velocities of the air-particles change from point to point. These differences of form depend upon the overtones associated with the fundamental. The quality of a sound, therefore, depends upon the number, the order, the relative intensity, and the relative phase of the overtones. A change in either of these particulars changes the complex law in accordance with which each particle of air is moving.

132. Consonant Intervals.

Whether two tones are consonant or dissonant must be determined entirely by the musical ear. The more consonant intervals are:

Unison (1:1), octave (1:2), octave + fifth (1:3), double octave (1:4), fifth (2:3), fourth (3:4).

Less consonant intervals are: the major third (4:5), the major sixth (3:5), minor third (5:6), and minor sixth (5:8).

The second and the seventh are always dissonant. Consonant intervals can be recognized much more readily than dissonant ones.

133. IMPERFECT CONCORD EXPLAINED BY BEATS. (Desch., 944.)

In order that two notes may produce distinct beats between their fundamentals they must be nearly in unison, for interference requires waves of very nearly the same length and amplitude. In seeking therefore for the elements that introduce dissonance by their beats, we must select intervals differing but little from unity. "For a given frequency of beats the harshness of the effect increases with the nearness of the notes to each other on the musical scale." The larger the number of such beating pairs that can be found among the overtones of an interval the less perfect the consonance. Take, for example, the C of 256 and E of 320 vibrations per second, constituting a true major third (4 : 5); and suppose that the first five overtones are associated with the fundamentals. The first six multiples of 4 are

64 $\overset{256}{4}$, 8, 12, $\overset{1024}{16}$, 20, 24.

The first six multiples of 5 are

64 $\overset{320}{5}$, 10, 15, 20, 25, 30.

The only multiples differing by unity are

4 and 5; 15 and 16; 24 and 25.

The first pair do not produce effective beats because their ratio is too large. But the ratics $\frac{16}{15}$ and $\frac{25}{24}$ are nearer unity and the corresponding tones will beat 64 times per second, since 256 and 320 are 4 and 5 times 64 respectively. If we examine a true fifth in the same way we shall find multiples differing by unity only among those whose ratio departs widely from unity. The major third has therefore more dissonant elements than the fifth.

134. RESULTANT TONES. (A. & B., 387; D., 426.)

Resultant tones are due to secondary waves, having their origin in the primary. They are either *difference* tones or *summation* tones. Difference tones have vibration-frequences equal to the difference between those of the primaries; summation tones are due to vibration-frequencies equal to the sum of those of the primaries.

Difference tones were discovered by Sorge in 1745 and independently by Tartini in 1754. They are known as Tartini's tones. Summation tones were discovered by von Helmholtz, first theoretically and then experimentally. Beats may occur between resultant tones as between primaries or overtones; and the consonance of any chord depends partly upon these secondary tones. Take, for example, a major triad with the first overtone of c′ and e′. The difference tones give us the following series:—

C....c.....g.....c′.....e′.....g′.....c″.....e″
64...128...192...256....320...384....512....640
1×64..2×64..3×64..4×64..5×64..6×64..8×64..10×64

No other tones can be derived by differences from c′, e′, g′, and c″, e″. The entire series consists of C with its harmonics up to the tenth. If we include g″, the resultant tone added will be due to 7 × 64 vibrations, not corresponding with any note in the scale. The absence of discordant intervals explains the pleasing effect of major triads.

Dal 14th

LIGHT.

PROPAGATION OF LIGHT.

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135. Nature of Light.

We may assume that light as a sensation is due to a mechanical action on the retina; the undulatory theory, now universally accepted, assigns this action to a disturbance propagated from the source by a wave-motion in a medium called the ether. Light as an objective phenomenon " now takes its place along side of electric phenomena, as but one of the forms of energy associated with that wonderful kind of matter provisionally called the ether."

When once Newton's corpuscular theory had failed because of its complexity and the crucial test applied to it by Foucault (article 142), the discovery of the time element in the transmission of light demonstrated the existence of an unknown medium. For the transmission of light is the transmission of radiant energy; and a medium of transmission is an absolute necessity as a repository of this energy during the time of transmission. Physical optics, therefore, consists of a study of optical phenomena as a wave-motion propagated in accordance with the principle of Huyghens (113).

136. Rectilinear Propagation of Light.

Simple experiments serve to show that light travels in a homogeneous medium along straight lines. A ray of light is only the direction in which any part of a wave-front is traveling, or the disturbance which is confined to this direction. The right line distribution of light is dependent upon the uniformity of the medium, so that the wave-surfaces shall be truly spherical about the source as a center. In such a medium the disturbance at any point is due almost entirely to the motion transmitted along a perpendicular dropped upon the wave-surface in some prior position. The secondary waves from all other points of the surface, as new centers of disturbance, can readily be shown to neutralize one another along this perpendicular on account of the excessive minuteness of the wave-lengths of all visible radiation. In media of unequal speed of transmission in different directions the wave-surfaces are not spherical, and a ray of light is no longer normal to the wave-surface.

137. IMAGES PRODUCED BY SMALL APERTURES. (T., 30.)

If light from a large luminous surface passes through a small opening of any shape in the shutter of a darkened room, and is received upon a white screen at some distance, it will form an image of the source. Each point of the luminous surface produces on the screen an image of the opening; but these images, infinite in number, are arranged so as to overlap one another on a surface like the source. This image is both inverted and perverted because of transmission of light along right lines. A small hole admits less light but produces a sharper image. With a larger opening and a converging lens more light is admitted and the image is sharper, because the light from any point of the luminous source is brought by the lens approximately to the corresponding point in the image.

138. THEORY OF SHADOWS. (T., 26; Desch., 952.)

An optical shadow is the region from which light is wholly or in part cut off by an opaque body. If the luminous source is a point, the shadow will be sharply defined; but when the source is a surface, the opaque body excludes all the light from a portion of the space behind it, called the *umbra;* and surrounding this is a region, called the *penumbra,* which is only partly screened from the luminous body. Fig. 48 shows how to determine the limits of both umbra and penumbra. The illumination diminishes gradually from the outer edge of the penumbra all the way to the total shadow. Partial solar eclipses occur in regions covered by the penumbra, and total eclipses within the umbra.

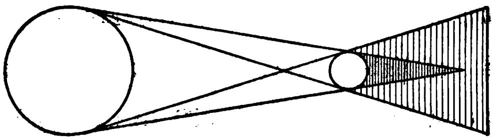


FIGURE 48.

139. SPEED OF LIGHT FROM JUPITER'S SATELLITES. (Desch., 960.)

The finite speed of light was discovered by Roemer in 1676 by means of the eclipses of preferably the inner satellite of Jupiter. These occur frequently and suddenly so that the interval between them is easily observed. Roemer discovered that the observed eclipses differ systematically from the computed times. When the earth is receding from Jupiter the interval between successive eclipses is longer than the mean, and the more rapid the recession the greater the excess; the reverse is true when the earth is approaching Jupiter. Roemer inferred a finite speed of light, so that when the earth is receding from Jupiter, light must require greater time at each eclipse to reach the earth. The sum of the excesses, which is the time required for light to pass across the earth's orbit, is about 16.5 min., and the speed deduced therefrom is about 186,500 miles per second.

140. BRADLEY'S METHOD FROM ABERRATION OF LIGHT. (T., 44; Desch., 961.)

The aberration of light was discovered by Bradley in

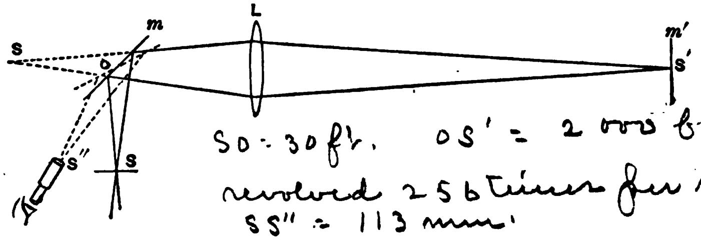
1728. It is the displacement in the apparent position of a star due to the orbital motion of the earth combined with the motion of light. It amounts to about 20.″5, and the tangent of this angle is the ratio of the earth's orbital speed to the speed of light. Since this tangent is about $\frac{1}{10,000}$, it follows that light travels 10,000 times faster than the earth in its orbit. Both of these astronomical methods depend upon a knowledge of the distance of the sun. The other two methods are terrestrial.

141. Fizeau's Direct Method. (Desch., 955.)

Suppose a toothed wheel, with breadth of teeth equal to spaces between, placed across a beam of light and revolving rapidly. The beam of light passes between two teeth to a distant mirror, and is reflected back on its path to the wheel. If during its passage out and back the wheel has revolved through the angular breadth of a tooth only, then the light on its return will be intercepted by a tooth; and the observer, situated on the same side of the wheel as the source, will fail to receive an image. If the speed of the wheel is doubled the image of the source will again come into view, and so on. It is evident that the speed of light in air is to that of the tooth, at the first disappearance, as twice the distance of the mirror from the wheel is to the breadth of the tooth. A careful repetition by Cornu in 1874 gave 298,500 kilometers as the speed in vacuo.

142. Michelson's Modification of Foucault's Method. (A. & B., 434; Desch., 957.)

Foucault's method was designed to determine the relative velocity of light through air and water as a crucial test between the emission and the undulatory theory of light. This method as modified by Michelson may be briefly described as follows: "Let S, Fig. 49, be a narrow slit, *m* a mirror which may revolve about an axis in its own plane, L a lens, and *m′* a second mirror. Light from a source behind S passes through the slit, falls on *m*, is reflected, when *m* is in a suitable position, through the lens L, and forms an image at S′. * * * * It is evident that any light reflected back from *m′* through L will return to the conjugate focus S, whatever the position of the mirror *m*, so long as it sends the light in such a direction as to pass through L both going and returning. If now the mirror *m* be given a rapid rotation clock-wise, light passing through L will return to find *m* in a changed position, and the image will be displaced from S to some point S″ to the left of S. Knowing the displacement S S″ and the number of rotations of the mirror per second, the time required for light to pass from *m* to S′ and back is determined." The speed of light determined by Michelson in 1882 is 299,853 kilometers per second. Professor Newcomb by a similar method obtained 299,860 ± 30 kilometers. When a tube with glass ends and filled with water is interposed between L and *m′*, the displacement of the image is increased, demonstrating that light travels slower through water than through air.



SO = 30 ft. OS′ = 2000 ft.
revolved 256 times per sec.
SS″ = 113 mm.

FIGURE 49.

REFLECTION AND REFRACTION.

143. EXPLANATION OF REFLECTION BY THE UNDULATORY THEORY. (T., 174; Desch.. 1102.)

Let A B, Fig. 50, be a plane wave-front, and A C the reflecting surface. Then if no obstacle prevented, when B reached C the wave-front would be D′ C. With A as a center and radius A D′ describe a circular arc. Then during the time the disturbance at B takes to reach C, the disturbance which had reached A will spread back into the medium as

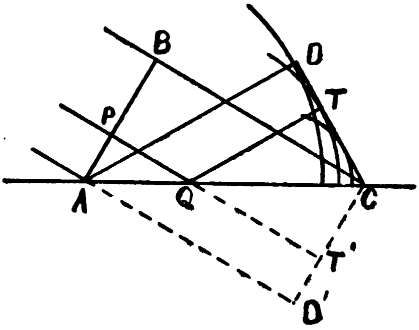


FIGURE 50.

a spherical wave for a distance A D′. The disturbance from P will reach Q and then diverge into a spherical wave with center Q and radius Q T′. All the circles which can thus be drawn ultimately intersect in the straight line C D. This is a section of the reflected wave-front. The equality of the triangles A B C and C D A proves the equality of the angles of incidence and reflection, the ray being perpendicular to the wave-front. Moreover, the incident and reflected rays lie in the same plane. The undulatory theory therefore furnishes are adequate explanation of the two laws of reflection.

144. IMAGES IN A PLANE MIRROR. (T., 60; Desch., 970.)

The image of a point in a plane mirror is found by drawing from the point a perpendicular to the mirror and producing it till the length is doubled. The extremity of the line is the image of the point, and rays proceeding from the point diverge after reflection as if they came from the image. Such an image is called *virtual* to distinguish it from a *real* image. Rays actual pass through a real image, while they only appear to come from a virtual one. The demonstration is the same as in sound (115), and depends upon the equality of the angles of incidence and reflection. As an object is made up of points, its image must be equal to it and symmetrically situated with respect to the mirror.

145. Images of Images. (Desch., 972; T., 63; G., 497; A. & B., 410.)

When light is reflected successively from two or more plane mirrors, the image in the first becomes the object for the second mirror; and the second image is found in precisely the same manner as the first one. So the second image may serve as object for a third, and so on, since in each case the light approaches any mirror as if it came from the virtual image in the next preceding one. To find the path of the light, for any image, from the object to the eye, draw a line from the eye to the image; and, from its intersection with the corresponding mirror, draw a line to the next preceding image; from the intersection of this line with the corresponding mirror draw a line to the next preceding image; and so on, till finally a line is drawn from the intersection with the last mirror to the object. The path of the rays will then be the portions of the lines so drawn which lie in front of the mirrors. The deviation sustained by a ray successively reflected from two mirrors is twice the angle between the mirrors. If the angle θ between the two mirrors is an aliquot portion of four right angles, the number of images, including the object, will be $\frac{2\pi}{\theta}$; and they will all be located on a circle, the center of which is on the intersection of the mirrors, Fig. 51. If the mirrors are parallel and face each other the number of images therefore becomes infinite.

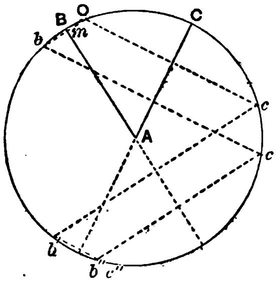


Figure 51.

146. DEVIATION PRODUCED BY THE ROTATION OF A MIRROR. (Desch., 976; T., 57; G., 499.)

If a plane mirror, on which a ray falls, be turned through any angle about an axis perpendicular to the plane of incidence, the reflected ray will be turned through twice that angle. This follows from the equality of the angles of incidence and reflection. For if the mirror is turned through any angle θ, its normal is turned through the same angle; hence the angles of incidence and reflection are both increased or decreased by this same angle, and therefore the angle between the incident and reflected rays is changed by $2\,\theta$. This principle is utilized in Hadley's sextant and in reading telescopes.

147. CONCAVE SPHERICAL MIRRORS. (Desch., 977; T., 64.)

Let A P B, Fig. 52, be a section of a concave spherical mirror by a plane passing through its center of curvature O, and let U be a luminous point. Then if a ray from U meets the mirror in P it will be reflected to V, so that the radius O P will bisect the angle U P V. Hence

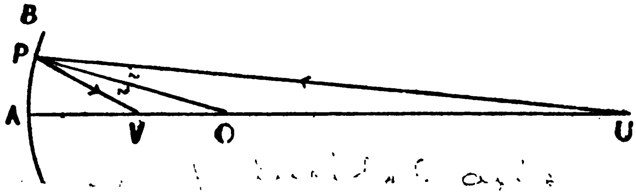


FIGURE 52.

$$\frac{VO}{VP} = \frac{UO}{UP}.$$

But if P is very near A, then A may be put for P in the equation, and

$$\frac{VO}{VA} = \frac{UO}{UA}.$$

If r is the radius A O, p represents U A, and p', V A, then substituting,

$$\frac{r - p'}{p'} = \frac{p - r}{p}, \text{ or } \frac{1}{p} + \frac{1}{p'} = \frac{2}{r}.$$

The point V is, to the degree of the approximation made, independent of the point P. U and V are called conjugate foci.

148. Discussion of the Formula and Principal Focus.

(1) As p increases p' decreases, and when p is infinite p' equals $\frac{r}{2}$. Hence parallel rays from an infinitely distance source come to a focus at a point midway between the center of curvation and the mirror. This point is called the *principal focus.*

(2) When p decreases p' increases, or object and image approach and meet at the center O, when $p = p' = r$.

(3) When p is less than r, and greater than $\frac{r}{2}$, p' is greater than r, or object and image have exchanged places.

(4) When p is less than $\frac{r}{2}$, p' is negative, the image is behind the mirror and therefore virtual. The two meet again at the mirror. Real images are on the opposite side of the center from the object and are therefore inverted. Virtual images are erect.

149. Caustics by Reflection. (T., 70.)

When the angular opening of the mirror is large then the approximations previously made are no longer admissible. Parallel rays do not meet accurately at the principal focus after reflection, but the image of a luminous point is then spread over a surface called a *caustic.* Let A P, Fig. 53, be a section of the mirror through its center O, and let

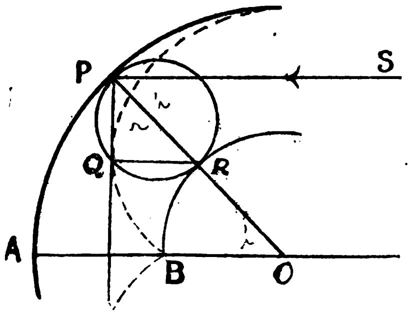


FIGURE 53.

S P be one of a system of rays parallel to A O. Then, since all the rays are symmetrical about A O, if we find the section of the caustic in the plane of the paper, the caustic surface may be found by revolving the figure about A O. Join P O and let P Q be the path of the reflected ray. Bisect P O in R and on P R as a diameter draw a circle; also with O as a center and with radius O R construct another circle R B. The two circles touch at R. Q P R equals R O B. But the arc Q R subtends Q P R at the *circumference*, and the arc P R subtends R O B at the *center* of a circle of double radius. Hence the arcs Q R and R B are equal; and if the small circle should roll on the inner circle the point Q would ultimately coincide with B, and would describe the epicycloid indicated by the dotted curve. Moreover, the point of contact R at any instant is fixed, and Q is therefore moving at right angles to Q R or in the direction of the reflected ray P Q. Hence all the reflected rays touch the epicycloid, which is therefore a section of the caustic surface; for the reflected rays cross everywhere on this section. At B is a cusp which is the principal focus of the mirror. Not all parallel rays after reflection pass through this focus. The effect of this inexactness upon the image is known as *spherical aberration.*

150. THE CONVEX SPHERICAL MIRROR.

The formula of article 147 is applicable to the case of a convex mirror. In this case the center O is at the left; and since distances to the right are positive, *r* is then negative. Therefore the formula becomes

$$\frac{1}{p} + \frac{1}{p'} = -\frac{2}{r}.$$

All the images in a convex mirror are virtual, since at least one of the distances p and p' must be negative.

151. The Undulatory Theory Applied to Refraction.

(T., 175; Desch., 1101; A. & B., 405; G., 612.)

Refraction is the change in direction of a ray of light in passing from a medium of one optical density to another. It depends upon the difference of the speed of light in the two media. In Fig. 54 let A C be the plane of separation of the two media, and A B a plane-wave in the first medium. If the medium remained the same, D′ C would be the position of the wave after the disturbance at B has traveled to C. But if the speed is less in the second medium, then with A as a center and a radius A D, equal to the distance traversed in the second medium while B C is described in the first, describe a circular arc. Then the sphere of which the arc is the trace will limit the extent of the disturbance propagated from A while that from B has traveled to C. In the same time the disturbance from P will pass to Q, and then from Q as a center it will spread out into the sphere whose radius Q T bears the same ratio to Q T′ that A D bears to A D′. All circles drawn in this way ultimately intersect along C D which is drawn through C tangent to the first circle. Hence C D is the new wavefront in the second medium. But the angles of incidence

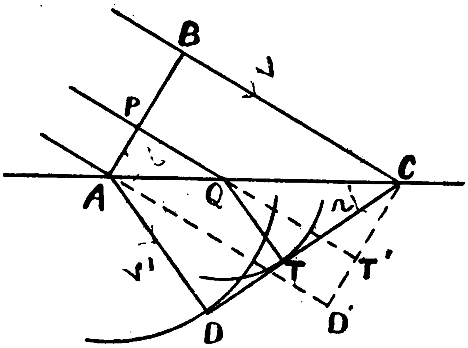


FIGURE 54.

and refraction are B A C and A C D respectively. Hence we have $\mu = \frac{\sin i}{\sin r} = \frac{\sin BAC}{\sin ACD} = \frac{BC}{AD} = \frac{v}{v'}$; or the index of refraction μ is a constant and equals the ratio of the speeds of light in the two media, since B C and A D are described in the same time. The undulatory theory, therefore, gives a satisfactory explanation of the laws of single refraction. If the first medium be a vacuum and the velocity in it be taken as the unit, then the absolute index for any medium will be the reciprocal of the speed of light in that medium.

152. NEWTONIAN THEORY OF REFRACTION. (Desch., 1102.)

The Newtonian theory ascribes the change in direction of a ray at the surface of separation between two media to the greater attraction of the denser medium for the corpuscles of light. The resultant of this attraction must be normal to the surface; therefore the component motion parallel to the surface remains unaffected when light enters a medium of different optical density. Let v and v' be the speeds of the corpuscles in the rarer and denser media respectively; and let i and r be the angles of incidence and refraction. Then the components of the speed parallel to the surface of separation will be $v \sin i$ and $v' \sin r$ respectively. Placing these equal to each other we obtain

$$\mu = \frac{\sin i}{\sin r} = \frac{v'}{v}.$$

But by the undulatory theory the index of refaction equals $\frac{v}{v'}$, which requires v to be greater than v'; the Newtonian theory makes v' greater than v. Foucault's experiment proved that the velocity in the denser medium is the smaller.

153. Refraction at a Plane Surface. (Desch., 1000; T., 85).

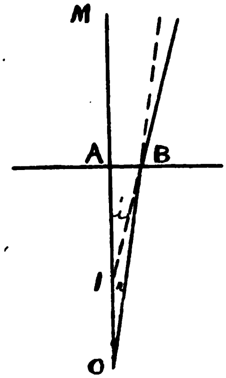


FIGURE 55.

Let M N, Fig. 55, be a normal to the surface A B; and let I B be an emergent ray from the the point O, and let I be the point in which it will meet the normal if produced downward. Then are B I A and B O A the angles of incidence and refraction, and

$$\mu = \frac{\sin B I A}{\sin B O A} = \frac{B O}{B I}.$$

But when B O is nearly normal, and the pencil of rays is only slightly divergent, B O is ultimately equal to A O, and B I to A I. Hence A I is equal to $\frac{A O}{\mu}$. If the eye is situated on the normal line an object at O will appear to be at I. Since μ for water is about $\frac{4}{3}$ and for glass $\frac{3}{2}$, an object in water cannot appear to be lower than $\frac{3}{4}$ its real depth below the surface, and in glass not more than $\frac{2}{3}$. Viewed obliquely the depth will appear still less. Conversely an eye under water cannot see an object at less than $\frac{4}{3}$ its distance from the surface.

154. The Critical Angle. (Desch., 997; T., 86; G., 515.)

When light passes from one medium into another which is less refracting, as from water into air, then the angle in the first medium is less than in the second. When the ray of light in the second medium passes along the surface, making an angle of 90° with the normal, the angle in the first medium is a maximum. This angle is called the *critical angle.*

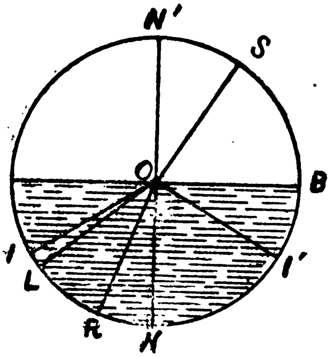


FIGURE 56.

Thus in Fig. 56 the ray R O in water emerges in air in the direction O S, and for the ray L O the direction of emergence will be O B, coincident with the bounding surface. If a ray I O makes a still greater angle with the normal than L O, then it cannot emerge into the second medium, but suffers *total internal reflection*, taking the direction O I′. The angle L O N is the critical angle. It may be computed as follows:

$$\mu = \frac{\sin 90^\circ}{\sin x}; \text{ whence } \sin x = \frac{1}{\mu},$$

or, *the sine of the critical angle is the reciprocal of the index of refraction.* Hence the greater the refractive index of any substance the smaller will be the critical angle. For air and water it is about 48° 30′; for glass it varies from 38° to 41°.

155. GENERAL CONSTRUCTION FOR REFRACTION OF A WAVE AT A SINGLE SURFACE. (D., 120.)

156. CONSTRUCTION FOR REFRACTION OF A SPHERICAL WAVE AT A PLANE SURFACE. (D., 122; Phil. Mag., Vol. 28, pp. 237, 238.)

[illegible] Sec 69

157. CONSTRUCTION FOR DEVIATION. (Desch., 1008; A. & B., 408.)

158. REFRACTION THROUGH A PRISM. (G., 519; Desch., 1007; T., 96; A. & B., 407; D., 477.)

A prism for optical purposes consists of a transparent medium bounded by two planes enclosing an angle which is less than twice the critical angle for the substance. This angle A, Fig. 57, is called the refracting angle. Since light is bent toward the normal on entering a medium of higher refrangibility, and away from it when passing into one of lower refrangibility, the path of a ray of homogeneous light through the prism may be such as L I E O. I is the point of incidence and E of emergence. Then the deviation of the emergent from the incident ray is

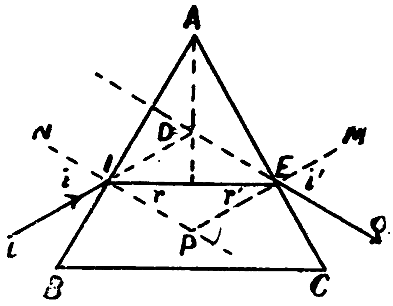


FIGURE 57.

$$D = i - r + i' - r' = i + i' - (r + r'.),$$

Since A equals the angle between the normals which equals $r + r'$,

$$A = r + r'.$$

If now the path of the ray through the prism is symmetrical with respect to the two faces, which is the condition of minimum deviation, then $i = i'$ and $r = r'$. Hence $A = 2\,r$, and

$$D = 2\,i - A, \text{ or } i = \frac{A + D}{2}.$$

Therefore, $\mu = \dfrac{\sin i}{\sin r} = \dfrac{\sin \frac{1}{2}(A + D)}{\sin \frac{1}{2} A}.$

For an approximate formula make the sines equal the angles, and

$$\mu = \frac{A + D}{A} = 1 + \frac{D}{A},$$

or

$$(\mu - 1)\,A = D.$$

159. REFRACTION AT SPHERICAL SURFACES. (T., 107; Desch., 1017; A. & B., 415.)

Let O, Fig. 58, be the center of curvation of the spherical surface A B. Let U be the point-source of homogeneous light, P the point of incidence, and let the refracted ray

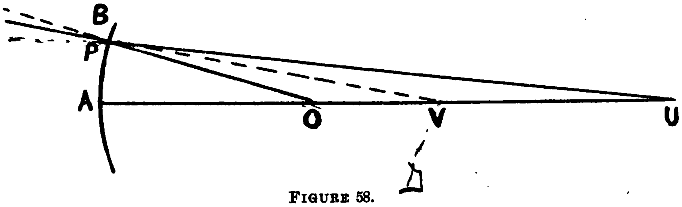


Figure 58.

produced backward be P V. Then

$$\sin UPO = \mu \sin VPO,$$

or

$$\frac{OU}{PU} = \mu \frac{OV}{PV}.$$

If the breadth of the surface is small in comparison with its radius of curvation, then we may write A for P to the first degree of approximation.

Hence

$$\frac{OU}{AU} = \mu \frac{OV}{AV}.$$

Using the same notation as in article 147,

$$\frac{p - r}{p} = \mu \frac{p' - r}{p'},$$

or

$$\frac{\mu}{p'} - \frac{1}{p} = \frac{\mu - 1}{r} \ldots\ldots (1).$$

If we put $\mu = -1$ this becomes the formula for the concave mirror.

If the ray now meets with another spherical surface, very near the first, with its center of curvation on the line O A, it will undergo another refraction. Then the distance p' becomes that of the luminous point from the second surface; and if we put q for the distance corresponding to p' for the first surface, and r' for the radius of the second surface we obtain

$$\frac{\frac{1}{\mu}}{q} - \frac{1}{p'} = \frac{\frac{1}{\mu} - 1}{r'},$$

$$\text{or } \frac{1}{q} - \frac{\mu}{p'} = \frac{1 - \mu}{r'} \ldots\ldots (2).$$

Adding (1) and (2) we have

$$\frac{1}{q} - \frac{1}{p} = (\mu - 1)\left(\frac{1}{r} - \frac{1}{r'}\right).$$

This is the approximate formula for a very thin *lens.*

160. Principal Focus and Discussion of the Formula. (T., 108.)

If the source is at an infinite distance the rays are parallel, p equals infinity, and the formula becomes

$$\frac{1}{q} = (\mu - 1)\left(\frac{1}{r} - \frac{1}{r'}\right).$$

If q for this case is represented by f, then

$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{r} - \frac{1}{r'}\right).$$

The distance f is called the *principal focal length.* For $\mu > 1, f$ is positive when $\frac{1}{r} - \frac{1}{r'}$ is positive, distances to the right being reckoned positive. The difference of the two curvatures will be positive for all lenses thinner at the center than at the edges. For such the principal focus is on the same side as the source, and parallel rays *diverge* after passing through the lens.

For all lenses thicker at the middle than at the edges $\frac{1}{r} - \frac{1}{r'}$, is negative, or the principal focus and the source are on opposite sides of the lens. Such lenses render parallel rays *convergent* after passing through them.

Finally the formula may be written

$$\frac{1}{q} - \frac{1}{p} = \frac{1}{f}.$$

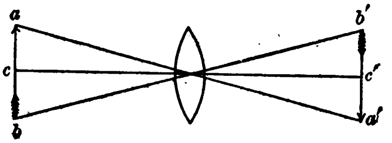
For diverging lenses f is positive; and since p is essentially positive, q is necessarily positive; that is, the object and image are on the same side of the lens and the image is therefore virtual.

For converging lenses f is negative. Therefore q may be either positive or negative according to the value of p.

The image may then be either real or virtual. When *p* equals *q*, we have

$$\frac{2}{p} = \frac{1}{f}, \text{ or } p = 2\,f.$$

Object and image when equidistant from the lens are distant from each other by a quantity 4 *f*. They cannot approach nearer than this for a real image.



161. Optical Center of a Lens. (Desch., 1015; A. & B., 421.)

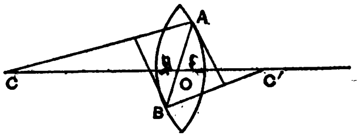


FIGURE 59.

Let C and C′, Fig. 59, be the centers of the two spherical surfaces of the lens. Draw any two parallel radii A C, B C′. Then the tangents at A and B are also parallel; and a ray incident at A passes through the lens without deviation, as if through a plate with parallel sides. Let A B be the path of such a ray, cutting the axis in O. Then A C O and B C′ O are similar triangles, and

$$C\,O : C'\,O :: C\,A : C'\,B.$$

$\frac{C\,O}{C\,O'}$ is therefore a constant, and all such lines as A B cut the axis at the same point O. This point is the *optical center.* Its distances from the two surfaces are directly as their radii. In plano-convex and plano-concave lenses it lies on the convex or concave surface.

162. Construction for a real Image in a Converging Lens. (A. & B., 422; Desch., 1020.)

To find the image of any point of an object it is necessary to trace the path of two rays only and find where they meet after refraction. All other rays from the object meet approximately at the same point. The two rays traced are, one along a *secondary axis* through the optical center, and the other parallel to the *principal axis* of the lens, which is the line joining the centers of curvation of the two surfaces.

In Fig. 60, A B is the object. From A draw A O through the optical center. Let F be the principal focus, and draw A L parallel to O F; after refraction it will pass through F and will intersect the first ray drawn at A′. Hence A′ is the focus conjugate to A. Other points of the image may be located in the same way.

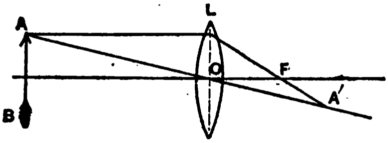


FIGURE 60.

A virtual image in either a converging or a diverging lens may be drawn in a similar manner.

DISPERSION.

163. DECOMPOSITION OF WHITE LIGHT. (G., 535; Desch., 1059; A. & B., 453.)

When a thin beam of sunlight is made to pass through a prism, it not only suffers deviation, as already pointed out, but it is resolved into a number of colors of different refrangibility. This phenomenon is known as *dispersion.* The red is the least refrangible color, and violet the greatest. The order of colors is red, orange, yellow, green, blue, violet. Another color, indigo, is sometimes located between the blue and violet.

Such a succession of colors in the order of refrangibility, obtained from any source of compound light, is called a *spectrum.*

Since the refractive index for light depends upon the retardation which it undergoes on entering an optically denser medium, it is evident that violet light suffers a greater retardation than red; or in other words, violet light travels slower than red through dense media. Measurements of wave-length show that the ethereal undulations producing extreme violet are the shortest of all those lying within the visible spectrum. It follows that disturbances of short wave-length undergo greater diminution of speed on entering dense transparent media than those of longer period.

164. DISPERSIVE POWER. (Desch., 1081; A. & B., 454.)

If two prisms of different materials are made with such angles that they give the same minimum deviation for the brightest part of the spectrum, it will be found that the lengths of the two spectra will not be the same. The angular separation of the colors varies with the transparent medium employed. If d', d'', d, represent the two extreme and the mean deviations and μ', μ'', μ, the corresponding indices of refraction, then the ratio

$$\frac{d' - d''}{d} = \frac{A\,(\mu' - 1) - A\,(\mu'' - 1)}{A\,(\mu - 1)} = \frac{\mu' - \mu''}{\mu - 1}$$

is called the *dispersive power* if the refracting angle A is small. It is the ratio of the difference of deviations of the extremities of the spectrum to the mean deviation. This ratio is different for different substances.

165. CHROMATIC ABERRATION. (Desch., 1080; T., 113.)

Since the homogeneous constituents of white light have different indices of refraction, it follows from the formula

$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{r} - \frac{1}{r'}\right)$$

that a simple lens has different focal lengths for different colors, and that f is less as μ is greater. Hence the violet light comes to a focus nearer the lens than red, the intermediate colors lying between these two. If therefore a screen be placed at the focus for violet the image of a white object will be colored red on the borders; if at the focus for red it will be bordered with violet; and a colorless image cannot be obtained with a single lens. This confusion of colored images is called *chromatic aberration.*

166. CONDITIONS OF ACHROMATISM. (T., 114; Desch., 1081; A. & B., 455.)

It will be evident from a consideration of section 164 that by varying the refracting angles of two thin prisms of different materials, such as crown and flint glass, and by combining them with their sharp edges turned in opposite directions, we may secure *deviation without dispersion* or *dispersion without deviation.* The conditions governing the first are those required to secure an achromatic image with two lenses; those of the second apply to a direct vision spectroscope.

The product of dispersive power, $\frac{A(\mu' - \mu'')}{A(\mu - 1)}$, by deviation, $A(\mu - 1)$, (section 158), is $A(\mu' - \mu'')$, or the dispersion. An achromatic combination of prisms or lenses requires that the dispersion produced by the two elements of the combination shall be the same for the two prisms or lenses. Of the two lenses necessary to form an achromatic pair one must be convex and the other concave. The dispersive power of ~~crown~~ flint glass is nearly twice as great as that of ~~flint~~ crown glass.

If the dispersion for two colors be made the same for any two media, in general the dispersion for the other colors will not be the same. While the order of the colors for the two spectra is the same the relative spaces occupied by the different colors are not the same. This is known as the *irrationality of dispersion.* Hence in general only two colors can be reunited by a combination of two lenses. Achromatism is therefore necessarily imperfect.

167. DARK LINES IN THE SOLAR SPECTRUM.

When a pure solar spectrum is obtained, in which adjacent colors are not allowed to overlap, it is found to be crossed by numerous dark lines. Fraunhofer studied these lines in 1814–15 and counted about 600 of them, marking the places of 354 upon a map of the spectrum. Hence they are often called Fraunhofer lines. To make them as distinct as possible the slit must be narrow and an achromatic lens must be placed at its focal distance from the slit, so that the light emerging from it may consist of parallel rays. After receiving these upon a prism adjusted for minimum deviation, the resulting spectrum may be viewed by an achromatic telescope. Such an apparatus constitutes a *spectroscope;* when it is provided with a divided circle for measuring deviations it is called a *spectrometer.*

168. THREE KINDS OF SPECTRA.

Three kinds of spectra may be distinguished, each of which may have several subdivisions:—

(1) *Bright line spectra.* The spectra of incandescent gases and vapors consist of a limited number of bright lines, each of which is a monochromatic image of the slit.

(2) *Continuous spectra.* Incandescent solids and liquids give continuous spectra. They exhibit a perfectly continuous succession of colors from one extremity to the other without any interruptions or gaps.

(3) *Absorption spectra.* These are discontinuous and are rendered such by losses due to absorption in the passage of light through transparent media. The solar spectrum is of this class, the absorption taking place chiefly in the outer envelope of the solar atmosphere, but to some extent also in the earth's atmosphere. These latter interruptions of continuity are known as *telluric* lines. Many gases, liquids, and solids produce discontinuity in the spectrum of light transmitted through them by means of their power of selective absorption.

INTERFERENCE AND DIFFRACTION.

169. Interference of Light from Two Similar Sources. (T., 185; A. & B., 436; D., 486; G., 617.)

The phenomena resulting from the superposition of two systems of waves of homogeneous light, traveling in very nearly the same direction, are called *interference.* Two such systems may be superposed in such a way that the disturbance of the one may annul that of the other. The essential condition is that the two wave-systems shall differ in phase by half a period. The most simple arrangement to exhibit interference was devised by Fresnel. B C D, Fig. 61, is an isosceles glass prism with the angle at C nearly 180°. Let O be a source of homogeneous light. The light passing through the prism will consist of two parts diverging from the virtual sources O_1 and O_2. Since the point A is equidistant from these sources the two systems will arrive at A in the same phase and will reenforce each other. Hence on the screen at A there will appear a bright band parallel to the edges of the prism.

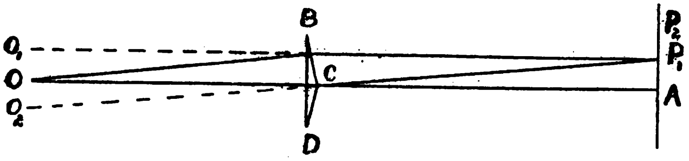


Figure 61.

Let P_1 be so situated that the distance $P_1 O_2$ shall exceed the distance $P_1 O_1$ by half a wave-length for the light employed. Then the two systems will meet at P_1 in complete opposition of phase, and interference will take place; so that through P_1 will be a dark band parallel to the bright one through A.

At P_2, where $P_2 O_2$ exceeds $P_2 O_1$ by an entire wave-length, there will be a bright band again; at some point P_3, where the two distances from O_1 and O_2 differ by one and a half wave-lengths, there will be another dark band;

and so on. The screen will therefore be illuminated by a series of bright bands alternating with dark ones. By measuring the distance from A to the first bright band and the distance A O, it is possible to compute the difference between $P_1 O_1$ and $P_1 O_2$ or to measure the wave-length of the light. A rude measurement of this kind shows that the wave-length for yellow light is only about $\frac{1}{50000}$th of an inch. Hence the vibration-frequency of yellow light is about 500 millions of millions per second.

170. INTERFERENCE PRODUCED BY THIN FILMS. (A. & B., 441; T., 195; D., 489; G., 624.)

Interference phenomena are produced by thin transparent films, due to the interaction of two wave-systems reflected from the two surfaces of the film. Let A A and B B, Fig. 62, be the parallel surfaces of the film. Light incident on the upper surface is partly transmitted and partly reflected. The transmitted portion is in part reflected from the second surface and emerges from the first surface along the same path with light which has undergone only a single reflection. If now the time required to traverse the film twice be the period of vibration of the homogeneous light employed, then the system undergoing internal reflection in the film will, so far as this reason is concerned, be in the same phase on emergence as the system reflected externally. But of the two reflections, one takes place in the rarer medium next to the denser, and the other in the denser next to the rarer. Hence one of the two interfering systems loses half an undulation relative to the other in the mere act of reflection. When therefore the thickness of the film and the angle of incidence are such that one ray falls behind the other by any whole number of wave-lengths, interference takes place with extinction of light; since in each case a difference of phase

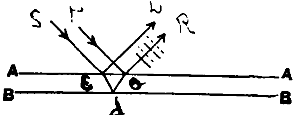


FIGURE 62.

of half a wave-length must be added because of reflection under opposite conditions at the two surfaces. With white light the extinction of one component by interference leaves colored fringes.

171. Diffraction Fringes with Narrow Apertures. (Desch., 1111; A. & B., 443; G., 620.)

When a beam of sunlight is admitted through a very narrow slit into a darkened room, and is received upon a screen at some distance, there will be seen a central band of white light in the direct path of the beam, bordered with colored fringes. Through so small an opening light does not pass as a definite ray, but diverges in all directions. Let *a b*, Fig. 63, be a section of the slit, the length of which is perpendicular to the plane of the paper, and let M N be the screen. Then *r* in the direction of the beam is nearly equidistant from every point between *a* and *b* of the slit; and all the disturbances starting from the several centers along *a b* arrive at *r* in nearly the same phase. Hence they conspire to produce maximum illumination at that point. Let *s* be a point so situated that the distances *a s*, *b s* shall differ by one wave-length for some color, as red. Then red will suffer extinction by interference at *s*. For if *a b* be divided into two equal parts, then the difference in distance of *a* and *d* from *s* is half an undulation; the elementary waves from *a* and *d* meet at *s* in opposite phases; and every wave from a point between *a* and *d* meets at *s* a wave in the opposite phase from a corresponding point between *d* and *b*. Total extinction of light of this particular wave-length takes place therefore at *s*.

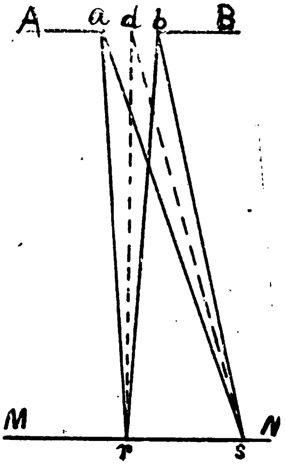


Figure 63.

If *a* and *b* are distant from a point on the screen by one and a half undulations, then the slit may be divided into three equal parts; the elementary waves from two of these parts interfere at the screen as explained, while those from the third part produce illumination. Hence with monochromatic light alternate illuminated and unilluminated positions of *s* may be found by moving the point to the right or left of *r*. These positions are those of maximum and minimum illumination. With white light extinction for different colors will take place at different distances of *s* from *r*, and hence colored fringes will appear on the screen.

172. SPECTRUM BY A DIFFRACTION GRATING. (A. & B., 446; Desch., 1113; G., 622; T., 190.)

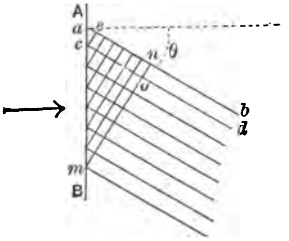


FIGURE 64.

A screen with several thousand parallel and equidistant apertures to the inch is called a *grating*. Suppose plane waves approach the grating in the direction of the arrow, Fig. 64. Let *a*, *c*, etc., be the parallel apertures, and let the parallel lines *a b*, *c d*, etc., be drawn at such an angle that the distance *a e* to the foot of the perpendicular from *c* upon *a b* shall equal some definite wave-length of light λ. Thus *a n* will be an exact number of wave-lengths $n\ \lambda$, *c o* will contain one wave less, and so on. The line *m n* will therefore touch the front of a series of elementary waves in the same phase. The obliquity of the lines may be increased till *a e* equals $2\ \lambda$, $3\ \lambda$, etc., and the result will be the same. All the transparent intervals in the grating send light to the focus in the same phase, if the wave-length is $\lambda = d \sin \theta$, in which *d* is the distance from center to center of the openings. For light of any other wave-length this coincidence in phase will not exist. If, for example, the difference between λ and $d \sin \theta$ is $\frac{1}{100} \lambda$, then the wave from the first opening will be in opposite phase to that from the fifty-first, that from the second will be in opposition to that from the fifty-second, etc. Hence all the light from the first fifty openings will exactly neutralize that from the second fifty in the direction assumed. Since the number of lines on a grating is very large, light of only one wave-length is found in any definite direction; different wave-lengths will be found in directions making different angles with the incident beam; so that a *pure* or *normal spectrum* is produced in which the angular separation of the different colors depends solely on their wave-lengths. From the formula $\lambda = d \sin \theta$ it is seen that the longest waves are found at the greatest deviations; (and for the first spectrum the wave-lengths are nearly proportional to the deviations.) Hence there is no irrationality of dispersion in a diffraction spectrum. For deviations at which $2 \lambda = d \sin \theta$ the spectrum is of the second order; for $3 \lambda = d \sin \theta$, of the third order, etc. To measure wave-lengths it is only necessary to be provided with a spectrometer to measure the angles θ, the distance d of the grating being known.

A reflecting grating differs from a transparent one only in having the fine parallel rulings on a polished surface of speculum metal. Professor Rowland's famous gratings are concave.

173. WAVE-LENGTHS AND VIBRATION-FREQUENCIES.

The unit employed in measuring wave-lengths of light is the *tenth-meter*, of which 10^{10} are required to make a meter. The following are the values for the principle Fraunhofer lines :—

A	7594	E	5270
B	6867.5	F	4861
C	6563	G	4293
D_1	589[illegible] 6	H_1	3968
D_2	5890	H_2	3933

Taking the speed of light as 300 million meters per second, or 300×10^{16} tenth-meters, the vibration-frequencies may be found by dividing this speed by the above values of λ. Thus the number of waves entering the eye per second are,

For A .... 395 millions of millions (395×10^{12}).
" D_2 ....509 millions of millions (509×10^{12}).
" H_2 ....762 millions of millions (762×10^{12}).

The visible spectrum lies between wave-lengths about 7500 and 3900 tenth-meters. Rowland has measured them from 7714.657 to 3094.736 tenth-meters, and has photographed them from about 7000 to 3000. Langley has measured lunar radiations with wave-lengths of 170,000 tenth-meters, or nearly twenty-three times as long as the longest waves exciting vision. The invisible spectrum extends several times the length of the visible spectrum beyond the extreme violet; so that the entire invisible spectrum actually explored is perhaps thirty times as long as the visible one. Physically the only difference existing among these radiations is one of wave-length. All of them represent energy which is converted into heat when absorbed by the proper surfaces; and perhaps all may be able to excite or precipitate chemical changes if the proper substances are found for different parts of the spectrum. The mechanism of the eye limits its receptivity to the visible spectrum. The differences formerly supposed to exist between the so-called light, heat, and actinic rays are therefore differences in the receptive apparatus.

COLOR.

174. Modes of Producing Colors from White Light.

Color has no objective existence; it is entirely subjective. The only physical differences corresponding to different colors are inequalities of wave-lengths. Red is due to the longest ether waves exciting vision and violet to the shortest. Hence the production of colored light from white light involves some process of isolation of vibrations of certain definite periods. These processes are three in number:—

(1). Refraction.
(2). Interference.
(3). Absorption.

The separation of white light into its colored components by means of the first two processes has already been explained.

175. THE COLOR OF OPAQUE BODIES. (Desch., 1087.)

All bodies except those with highly polished surfaces reflect light by irregular reflection from greater or less depths within the body. If all the components of white light are reflected in equal proportion the body appears white or gray; but if the body exhibits any inequality in its relative absorbing and reflecting power for radiations of different wave-lengths in the visible spectrum, then it will appear colored, the color being due to the mixture of the rays which are reflected. A mixture of the absorbed radiations would produce a color *complementary* to that of the reflected radiations.

176. COLOR OF TRANSPARENT BODIES.

If a body is transparent to radiations of certain wave-lengths and not to others, it appears colored by transmitted light; and the color is due to the mixed impression produced by the transmitted radiations. The restriction to radiations within the visible spectrum again applies. If a piece of glass appears red by transmitted light it is because it transmits red and absorbs the other colors. Hence if glasses of various colors are placed across a beam of light, so that the light passes through them in succession, it will lose something at each passage, and the light emerging is that which is not absorbed by any, or that which is transmitted by all in common. This is a process of successive subtraction and not a mixture of colors by addition. It illustrates the mixture of pigments. Thus blue and yellow media usually both transmit green, and green is the color transmitted through the two together. So blue and yellow liquids, which do not react chemically, appear green by transmitted light when mixed together. The spectrum of light transmitted through colored media exhibits *absorption bands* which show what kind of light has been stopped.

177. Mixing Colors.

In order to perceive the mixed effect due to two or more colors it is necessary that they shall fall upon the retina either simultaneously or in quick succession. But very different mixture may produce the same visual impression. "Every color in nature can be exactly imitated by an infinite variety of different combinations of elementary rays." Thus "any one of the elementary colors, from the extreme red to a certain point in the yellowish green, can be combined with another elementary color on the other side of the green in such proportion as to yield a perfect imitation of ordinary white." The unaided eye can tell nothing about the composition of colored light. It must be studied by means of the spectroscope.

POLARIZED LIGHT.

178. Polarization. (D., 463.)

The vibrations of the ether in light are transverse to the direction in which the light is traveling.

If these vibrations are confined to a single plane the light is said to be *plane polarized.* If the vibrations are in circles about the ray as an axis, the light is *circularly polarized.*

If the ether rotates in an ellipse about the ray as an axis, the light is *elliptically polarized.*

The relation of circular and elliptic polarization to two systems of plane vibrations at right angles to each other will be readily understood from the composition of S. H. M's of the same period.

Common light is much more complex than polarized light. No single plane of vibration has any advantage, and it is probable that the vibrations in common light run rapidly through a great variety of circles, ellipses, and complex harmonic curves of every kind.

179. DOUBLE REFRACTION IN ICELAND SPAR. (D., 494; T., 205; A. & B.. 474; Desch., 1010.)

In the study of refraction thus far we have assumed one velocity in the denser medium and hence but one refracted ray for every incident ray. This is true only so long as the medium is isotropic. If a crystal of Iceland spar (crystalized calcium carbonate) is laid on a printed page, the letters will in general appear double; or if a thin beam of light is admitted through a round opening and passed through the spar, two discs of equal illumination will appear on the screen. Hence the incident beam is divided into two beams in the spar by the process of double refraction. The one refracted least is called the *extraordinary* ray and the other the *ordinary* ray.

The ordinary ray follows the laws of single refraction, and its refractive index for the D line is 1.658. The extraordinary ray does not in general lie in the plane of incidence, and its refractive index varies from 1.658 to 1.486. The minimum value is called the extraordinary index.

At two opposite corners of the crystal three obtuse angles meet. A line making equal angles with the planes of these obtuse angles is called the *optic axis* of the crystal. Any plane normal to a surface of the crystal and parallel to the optic axis is called a *principal plane.* If two corners are cut away by planes perpendicular to the optic axis, a beam incident normally on either of these surfaces does not suffer bifurcation.

180. Explanation of Double Refraction. (A. & B., 475; D., 496; T., 207; Desch., 1122.)

Since the refractive index is inversely as the speed of light in any medium, it follows that for doubly refracting substances there are two velocities of light. For the ordinary ray the velocity is the same in every direction, while for the extraordinary ray the velocity varies with the direction of the ray between limits proportional to 1.486 and 1.658. It has been found for Iceland spar that the phenomena can be completely represented by supposing that a disturbance, started in the interior of a crystal, gives rise to two concentric wave surfaces, one spherical and the other a flattened ellipsoid, with its polar diameter parallel to the optic axis, and equal to the diameter of the sphere. The polar and equatorial diameters are as 1.486 to 1.658. The two wave surfaces therefore touch at the extremities of the polar diameter, and the spherical surface of the ordinary ray lies wholly within the ellipsoid of revolution. A crystal in which the wave surfaces are thus related is called a *negative uniaxal* crystal. If the sphere encloses the ellipsoid, the two touching at the poles of the ellipsoid, the crystal is a *positive uniaxal* crystal.

From these two wave surfaces the path of the two rays may be determined by the method already employed in § 151. Thus let $i\ c$, Fig. 65, be the direction of the incident light falling on a crystal of Iceland spar, A B. Let the plane of incidence contain the optic axis $a\ b$. With c as a center and with $i\ c$ produced as the major axis, draw a circle and an ellipse with major and minor axes proportional to the ordinary and extraordinary indices of refraction, the

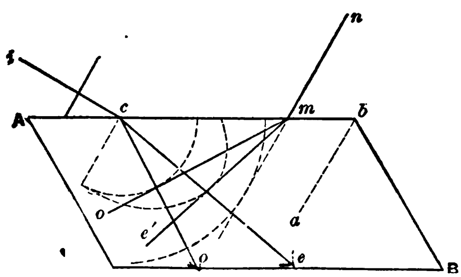


Figure 65.

radius of the circle being the semi-minor axis. Draw also a larger circle with a radius equal to the distance light will travel in air while the ordinary ray travels over the radius of the small circle in the spar. Then if *n m* is parallel to the incident wave, draw tangents from *m* to the inner circle and to the ellipse; the lines *c o*, *c e*, connecting *c* with the points of tangency, give the direction of the ordinary and extraordinary rays respectively.

It is evident that if the optic axis were perpendicular to the plane of the paper through *c* the section of the ellipsoid made by the plane of incidence would be a circle, the extraordinary ray would lie in the plane of incidence for all angles of incidence and would have its least refractive index.

181. Polarization by Double Refraction. (A. & B., 476.)

The examination of light transmitted by Iceland spar, either by another crystal or by a piece of unsilvered plate glass, exhibits a marked difference between it and ordinary light. Let the extraordinary ray be intercepted by a screen, and let the ordinary ray fall on the plate glass at an angle of incidence of 57°. If now the plane of incidence coincides with the principal plane of the spar, the light will be reflected like ordinary light; but if the mirror is rotated round the beam of light as an axis the reflected light will grow dimmer and dimmer; and when the plane of incidence is at right angles with the principal plane of the spar, the light will fail altogether. If the rotation is continued, the light gradually regains its maximum intensity at 180°, and again fails at 270°. The extraordinary ray exhibits the same peculiarities in the same order, but it has its maximum brightness at 90° and 270°, and fails at 0° and 180°. Both rays are plane polarized, the ordinary in the plane of the principal section, and the extraordinary in a plane at right angles thereto. The vibrations composing the ordinary ray are at right angles to the optic axis, while those of the extraordinary ray are in a plane containing the optic axis and the ray. Hence when the vibrations are parallel to the mirror the light is reflected.

182. POLARIZATION BY REFLECTION. (T., 213; A. & B., 480.)

When light has been reflected from such surfaces as water, glass, polished wood, etc., at a definite angle, which depends upon the nature of each substance, it is found to possess all the properties of light polarized by Iceland spar. Brewster discovered that if the reflected ray is at right angles to the refracted ray, it is completely polarized. In other words the angle of incidence is the polarizing angle when its tangent equals the index of refraction. For when the reflected and refracted rays are at right angles, the corresponding angles are complementary. Hence $\sin r = \cos i$, and

$$\mu = \frac{\sin i}{\sin r} = \frac{\sin i}{\cos i} = \tan i.$$

The ray transmitted by transparent media contains the same amount of polarized light as the reflected ray, and the planes of vibration of the two rays are at right angles.

183. THE NICOL'S PRISM. (A. & B., 481; Desch., 1123.)

A single beam of polarized light may be produced by transmission through a bundle of parallel plates; also by the passage through a plate of tourmaline (cut parallel to the axis), which absorbs one of the rays and transmits the other; but the most effective arrangement for the purpose is a Nicol's prism. This is made as follows: A large crystal of Iceland spar is cut through by a plane, the trace of which is A B, Fig. 66. The plane of the figure is a principal plane; the angle C B A is 22°, and B C A 68°. The two faces of the section after polishing are cemented together with Canada balsam, which has a refractive index intermediate between that of the ordinary and the extraordinary rays. Hence when a ray of light *a b* enters the

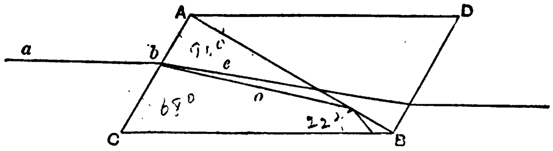


FIGURE 66.

prism it is divided into two rays, *b o* and *b e*; but *b o* meets the Canada balsam at an angle greater than the critical angle; and since it is passing from a medium of higher to one of lower index of refraction, it suffers total internal reflection and is absorbed by the blackened cork in which the prism is held. The other ray passes on through, since it has a lower index of refraction in the spar than in the balsam. The direction of vibration of the transmitted ray is the shorter diagonal of the end.

184. EXTINCTION OF LIGHT BY TWO CROSSED NICOLS.

When the light which has passed through one Nicol is made to fall on a second, the amount transmitted will depend upon the relation of the planes of vibration of the two. The first is called the *polarizer* and the second the *analyzer.* If their shorter diagonals are parallel, then the light from the polarizer will constitute the extraordinary ray in the analyzer and will pass on through unaffected. If now the analyzer is turned round on the ray of light as an axis, the transmitted beam will diminish in brightness, and will disappear entirely when the rotation has reached 90°. The Nicols are then said to be crossed, and the light entering the analyzer forms its ordinary ray, and is lost by internal reflection. In intermediate positions the rectilinear vibrations of the plane polarized ray are resolved into two rectangular vibrations in directions corresponding with those of the ordinary and the extraordinary rays for the analyzer. The ordinary ray is stopped by reflection, the extraordinary alone passing through. The intensity of the transmitted light is proportional to the cosine of the angle through which the analyzer has been rotated from the position of parallelism with the polarizer.

185. Effect of Interposing a Doubly Refracting Plate. (T., 237; A. & B., 483.)

Suppose plane polarized homogeneous light from the polarizer to fall on a thin doubly refracting plate of mica or selenite. Within the plate it will be divided into two rays unless one of the directions of vibration of the thin plate coincides with that of the polarizer. Let O x and O y, Fig. 67, be the two directions of vibration for the mica or selenite plate. Then if O A represents the semiamplitude of vibration for the incident ray, it will be resolved into two vibrations whose semiamplitudes are O M and O N. Both of these rays will pass through the plate unchanged except that one travels faster than the other, and therefore a difference of phase will result, dependent upon the thickness of the plate. The recombination of these two vibrations on emergence will result in elliptic motion, all the possible ellipses being inscribed within the rectangle A A′ B B′. The amplitudes of the motion parallel to O x and O y are not altered, but the maximum displacement in one direction is no longer simultaneous with that in the other, unless the difference of phase becomes some multiple of 2π. The light emerging from the thin plate is then in general *elliptically polarized.* If O M equals O N and the difference of phase becomes an odd multiple of $\frac{1}{2}\pi$, the light will be *circularly polarized.*

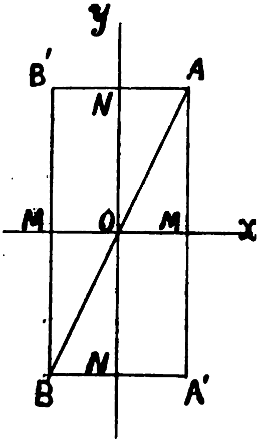


Figure 67.

Suppose now that the elliptically polarized light passes on to the analyzer, the two Nicols being crossed. The introduction of the mica plate in general restores the light. If the mica plate is made to rotate in its own plane the light vanishes for successive positions differing by a quadrant of rotation. In these positions the directions of vibration for the interposed crystal coincide with the principal planes of the Nicols; and the light from the first Nicol passes unchanged through the crystal and is extinguished by the second Nicol. Midway between these positions of extinction the light transmitted by the system is brightest. If the mica is of such thickness as to produce circular polarization, the rotation of the analyzer does not alter the brightness of the transmitted light. When the elliptically polarized ray enters the analyzer each of the two rectilinear components of the elliptical motion is resolved in the two directions of vibration for the analyzer. One pair of these components unite to form the ordinary ray, which is extinguished; the other pair form the extraordinary ray, which is transmitted.

186. Colors Produced by Polarized Light.

Of the two pairs of components of the elliptically polarized light entering the second Nicol, the one forming the extraordinary ray exhibit interference if the two components have opposite phases. The other pair then have the same phase; and if the analyzer is rotated through 90°, they form the extraordinary ray and are transmitted. If now the light is not homogeneous, then as the difference in phase depends upon wave-length, the retardation of one component as compared with the other is such as to produce complete interference for a definite wave-length and partial interference for waves of approximately the same length. The corresponding colors then suffer complete or partial extinction, while the remaining colors of the incident light are transmitted, forming a colored beam.

187. COMPLEMENTARY COLORS IN THE TWO STANDARD POSITIONS OF THE ANALYZER. (A. & B., 485.)

If the components along O x, Fig. 67, annul each other, the color to which they correspond is wanting in the plane polarized light vibrating in this direction; but at the same time the components along O y are added, and the same color is found undiminished in the light whose vibrations are confined to this plane. For other colors "the relative retardation is different; but for each vibration period, the component in the direction O x combined with that in the direction O y represents the total light for that period in the beam entering the analyzer." Hence the sum of the two represents all the light entering the analyzer; and therefore the light transmitted when the Nicols are crossed must be complementary to that passing when they are parallel, if the incident light is white.

If the doubly refracting plate is thick enough to produce a relative retardation of several wave-lengths for extreme violet, it will produce a retardation of half as many wave-lengths for red, and an intermediate number for intermediate colors. Hence with crossed Nicols extinction of these colors will take place. But the transmitted light will consist of the different colors in nearly the same proportion as in white light, and it will therefore be white but of diminished intensity.